CALCULUS I REVIEW SHEET

20130808

LIMITS AND CONTINUITY.

Definition. *Limit (informally)*

- $\lim_{x\to a} f(x) = L$ means as x gets "close" to a, $f(x)$ gets "close" to L.
- $\lim_{x\to a^-} f(x)$ is the limit from the left.
- $\lim_{x\to a^+} f(x)$ is the limit from the right.

note: don't care what happens to $f(x)$ at $x = a$

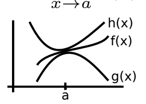
Remark. Can do limits numerically and graphically. Need some rules to do limits analytically.

Theorem. *limit rules*
If $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$,
then

- $\lim_{x\to a} x = a$
- $\lim_{x\to a} c = c$
- $\lim_{x\to a} (f(x) + g(x)) = L + M$
- $\lim_{x\to a} f(x)g(x) = LM$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ *for* $M \neq 0$
- $\lim_{x\to a} (f(x))^{\frac{1}{k}} = L^{\frac{1}{k}}$ *for* k *integer* > 0, $L \geq 0$

Remark. Two tricks for indeterminate limits: factoring and multiplying by conjugate

Theorem. *squeeze*
If $g(x) \leq f(x) \leq h(x)$ *and* $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$
then $\lim_{x\to a} f(x) = L$



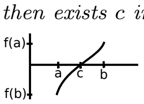
Definition. *infinite limit and limit at infinity*

- *infinite limit* means $\lim_{x\to a} f(x) =$ "$\frac{c}{0}$"
  i.e. $f(x)$ "blows up" at $x = a$
  note: to find out whether it is $-\infty$ or ∞, sample a point on each side
- *limit at infinity* means $\lim_{x\to\infty} f(x)$

Definition. *continuity*

- $f(x)$ *continuous* at $x = a$ means $\lim_{x\to a} f(x) = f(a)$
- $f(x)$ *continuous from left* at $x = a$ means $\lim_{x\to a^-} f(x) = f(a)$.
- $f(x)$ *continuous from right* at $x = a$ means $\lim_{x\to a^+} f(x) = f(a)$.
- $f(x)$ *continuous on interval* means continuous at each point on interval.
- $f(x)$ *discontinuous* at $x = a$ means not continuous.
  Three types: removable, infinite, jump

Theorem. *IVT*
If $f(x)$ *continuous on* $[a, b]$ *and* $f(a)f(b) < 0$
then exists c *in* (a, b) *such that* $f(c) = 0$



Definition. *limit (formally)*
$\lim_{x\to a} f(x) = L$ means for every $\epsilon > 0$ there exists a $\delta > 0$ such that (if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$)

DERIVATIVES.

Definition. *derivative*
The derivative of $f(x)$ at $x = x_0$ is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} = \frac{\mathrm{d}f}{\mathrm{d}x}(x_0)$$

intuition: derivative is the slope of the tangent line $g(x) = f'(x_0)(x - x_0) + f(x_0)$

Theorem. *relationship between continuity and derivative*

- *If* $f(x)$ *differentiable at* a
  then continuous at a
- *If any of these hold*
  (a) $f(x)$ *discontinuous at* $x = a$
  (b) $f(x)$ *has corner at* $x = a$
  (c) $f(x)$ *has vertical tangent at* $x = a$
  then $f(x)$ *is not differentiable at* $x = a$

Theorem. *derivative rules*

- $(c)' = 0$
- $(x^a)' = ax^{a-1}$
- $(cf(x))' = cf'(x)$
- $(f(x) + g(x))' = f'(x) + g'(x)$
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- $(f(g(x)))' = f'(g(x))g'(x)$
- $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$
- $(a^x)' = a^x \ln(a)$
- $(\log_a(x))' = \frac{1}{x \ln(a)}$
- $(\sin(x))' = \cos(x)$
- $(\cos(x))' = -\sin(x)$
- $(\tan(x))' = \sec^2(x)$
- $(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$
- $(\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$
- $(\arctan(x))' = \frac{1}{x^2+1}$

Remark. Can take derivatives of implicitly defined functions by taking derivatives of both sides and solving for $y'(x)$

Remark. A trick for doing derivatives with many powers is taking log, then derivative, then solving for $f'(x)$

DERIVATIVES AND GRAPHING.

Definition. *extrema*

- $\max_{(a,b)} f(x) = f(x_0)$ means $f(x_0) \geq f(x)$ for all x in (a, b).
- $\min_{(a,b)} f(x) = f(x_0)$ means $f(x_0) \leq f(x)$ for all x in (a, b).
- A *critical point* x_0 means $f'(x_0) = 0$ or DNE.

Remark. To find extrema of a function on an interval, check critical points and endpoints.

Theorem. *first derivative test for monotonicity*

- *If* $f'(x) > 0$ *on* (a, b),
  then $f(x)$ *increasing on* (a, b)
- *If* $f'(x) < 0$ *on* (a, b),
  then $f(x)$ *decreasing on* (a, b)
- *If* $f'(x) = 0$ *on* (a, b),
  then $f(x)$ *constant on* (a, b)
- *If* $f'(x)$ *DNE,*
  then $f(x)$ *has a discontinuity, corner, or vertical tangent*

Definition. *concavity*

- $f(x)$ is *concave up* means $f''(x) > 0$
  i.e. $f'(x)$ increasing
- $f(x)$ is *concave down* means $f''(x) < 0$
  i.e. $f'(x)$ decreasing
- An *inflection point* $x = x_0$ means $f''(x_0) = 0$ or DNE.
  i.e. where concavity changes

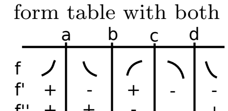
Theorem. *second derivative test for extremum*
Let $f(x)$ *have a critical point* $x = x_0$

- *If* $f(x_0)$ *concave up i.e.* $f''(x_0) > 0$
  then $x = x_0$ *is a local min*
- *If* $f(x_0)$ *concave down i.e.* $f''(x_0) < 0$
  then $x = x_0$ *is a local max*
- *If* $f''(x_0) = 0$
  then we know nothing

note: this is an alternative to sampling $f'(x)$ *left and right of* $x = x_0$

Remark. To sketch the graph of $f(x)$:

(1) determine critical points, monotonicity
(2) determine inflection points, concavity
(3) form table with both monotonicity and concavity



(4) draw vertical and horizontal asymptotes
(5) place extrema and inflection points
(6) sketch the graph

DERIVATIVES AND APPLICATIONS.

Remark. Strategy for optimization word problems:

(1) find function and domain of definition
(2) find critical points
(3) verify max or min
(4) answer original question

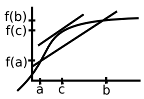
Definition. *linear approximation*

- The *linear approximation* to $f(x)$ around $x = a$ is the line

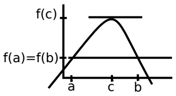
$$g(x) = \underbrace{f'(a)}_{\text{slope}}(x-a) + f(a)$$

- error=approx-exact
- % error=$\frac{\text{approx-exact}}{\text{exact}} \times 100$

Theorem. *Mean Value Theorem (MVT)*
If $f(x)$ continuous on $[a, b]$, differentiable on (a, b),
then exists c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
intuition: slope of secant line equals slope of some tangent line between a and b.



Theorem. *Rolle's (special case of MVT)*
If $f(x)$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$
then exists c in (a, b) such that $f'(c) = 0$.



Remark. Next is another trick for computing limits when plugging gives "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$". We couldn't state it earlier because it requires derivatives.

Theorem. *L'Hopital's Rule*
If $\lim_{x\to a} \frac{f(x)}{g(x)} =$ "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"
then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
note: might need multiple applications
note: also works for $x \to \pm\infty$, $x \to a-$, $x \to a^+$
trick: for "$\infty - \infty$" or "$0 \cdot \infty$", convert to "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

INTEGRALS.

Definition. *antiderivative*
An antiderivative of $f(x)$ is another function $F(x)$ such that $f(x) = F'(x)$
note: not unique; up to additive constant

Definition. *indefinite integral*
Let $f(x) = F'(x)$ i.e. $F(x)$ is antiderivative.
The *indefinite integral* of $f(x)$ is

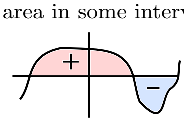
$$\int f(x)\mathrm{d}x = F(x) + c$$

Remark. Next are rules for simplifying complicated integrals.

Theorem. *integral rules*

- $\int (f(x) + g(x))\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x$
- $\int cf(x)\mathrm{d}x = c\int f(x)\mathrm{d}x$
- $\int 1\mathrm{d}x = x + c$
- $\int x^a\mathrm{d}x = \frac{x^{a+1}}{a+1} + c$ *for $a \neq -1$*
- $\int \frac{1}{x}\mathrm{d}x = \ln|x| + c$
- $\int e^{ax}\mathrm{d}x = \frac{1}{a}e^{ax} + c$
- $\int \sin(ax)\mathrm{d}x = -\frac{1}{a}\cos(ax) + c$
- $\int \cos(ax)\mathrm{d}x = \frac{1}{a}\sin(ax) + c$

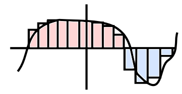
Remark. Intuitively, the integral is the area between a function $f(x)$ and the x-axis. Area below the x-axis is negative. We need a method to compute the area in some interval $[a, b]$.



Definition. *Riemann sum*
The *Riemann sum* of $f(x)$ over $[a, b]$ is the approximation of the area under the graph by summing areas of rectangles,

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{k=1}^{N} f(x_k)\Delta x$$

where heights of rectangles are $f(x_i)$ and width of each rectangle is $\Delta x = \frac{b-a}{N}$
note: can use various quadratures (left, right, midpoint, etc.)

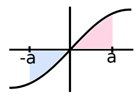


Definition. *Riemann integral*

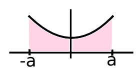
$$\int_a^b f(x)\mathrm{d}x = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\Delta x$$

Remark. Tricks using symmetry

- For odd functions $f(x)$, $\int_{-a}^{a} f(x)\mathrm{d}x = 0$



- For even functions $f(x)$, $\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x$



Theorem. *inequalities and integrals*

- *If $f(x) \leq g(x)$ on $[a, b]$,*
  then $\int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$
- *If $m \leq f(x) \leq M$ on $[a, b]$,*
  then $m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$

UNIFICATION OF INTEGRALS AND DERIVATIVES.

Theorem. *Fund Thm of Calc 1 (FTC1)*
If $F'(x) = f(x)$ on $[a, b]$,
then $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$.
Intuition: can compute definite integrals just from antiderivatives at endpoints!!!

Remark. Does every continuous function have an antiderivative? Need theorem.

Theorem. *FTC2*
If $f(x)$ continuous on $[a, b]$
then exists antiderivative $F(x)$ defined by $F(x) = \int_a^x f(t)\mathrm{d}t$ i.e. integrate both sides of $F'(x) = f(x)$.
note: this existence theorem is not useful in computing $F(x)$

METHODS FOR INTEGRALS.

Theorem. *taking derivative of integral*

- $\left(\int_a^x f(t)\mathrm{d}t\right)' = f(x)$
- *more generally:*
  $\left(\int_a^{g(x)} f(t)\mathrm{d}t\right)' = f(g(x))g'(x)$
- *most generally:*
  $\left(\int_{g_1(x)}^{g_2(x)} f(t)\mathrm{d}t\right)' = f(g_2(x))g_2'(x) - f(g_1(x))g_1'(x)$

Theorem. *reverse of chain rule*
If $f(x)$ has antiderivative $F(x)$
then $f(g(x))g'(x)$ has antiderivative $F(g(x))$.

Remark. u-substitution trick
The above theorem can be used to compute complicated integrals:
$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u$
where $u = g(x)$ and $\mathrm{d}u = g'(x)\mathrm{d}x$