Remark.
- combinatorics:
  - a maelstrom of isolated problems
  - a coherent subject with central themes, techniques, and results
- this book is for graduate level 2 semester course semseter one: nonoptional parts of ch 1-10, plus early portions of ch 12-14
  chapters after Part II are relatively independent except for background language from early parts
  see pg xi,xii for two other possible courses
- sections with "optional" or "*" can be skipped on first reading
  $(-)$ means easier problem, $(+)$ means more difficult problem, and $(\diamond)$ means particularly interesting or instructive
- Book organization
- part I is like undergrad courses in combinatorics, but concise and deaper..
  basic techniques involving bijective arguments, generating functions, recurrence relations, the inclusion-exclusion principle, signed involutions (an expansiton of inclusion/exclusion). A combinatorial point of view of Young Tableaux and Polya Redfield counting. Deep algebraic aspects are omitted.
- part II is like undergrad courses in graph theory, but concise and deaper.
  elementary graph theory with classical results. Large topics only mentioned in passing or exercises, including diameter, encodings, domination, decomposition, packings, genus, minors, nowhere-zero flows, generalized colorings, structure of special families. Other methods appear in later chapters
- part III considers questions about sets and order relations.
  Our most general structureal objects: families of sets, whigh generalize graphs to hypergraphs. Four aspects of set systems: Ramsey theory, extremal set theory, structural aspects of posets and matroids, and combinatorial designs. Omitted are many aspects of algebraic combinatorics
- part IV are methods applied questions that arose in parts I-III.
  methods from probability, algebra, and geometry – applied to questions about graphs and sets. Combinatorial applications to geometric questions. focus on methods and connections.

PART I: ENUMERATION

Remark. this part: undergrad combinatorics course, but concise and in-depth, and with connections among topics

1. COMBINATORIAL ARGUMENTS

Remark.
- exhaustive listing is naive and sometimes impractical
- combinatorial argument means "explicit counting argument" or bijection argument
  elementary but can be subtle
- combinatorial object

Remark. counting problem usually asks simultaneously for each value of parameter(s) (variable(s))
convention: universe of variables is what makes sense, usually $\mathbb{N}$ or $\mathbb{N}_0$
solution (for case one parameter n) is sequence $\langle a \rangle$ of values $\{a_n\}_{n=0}^{\infty}$
Types of solutions:
- formula expressing a_n as a function of other parameters
- an effective and fast procedure for computing individual terms
  eg (ch2) recurrence relation for $\langle a \rangle$ expresses a_n as a function of $a_0, ..., a_{n-1}$; "solved" by formula for a_n; need initial values to specify $\langle a \rangle$ completely
- (§2.3) asymptotic formula for a_n is formula $g(n)$ st $\frac{g(n)}{a_n} \xrightarrow{n\to\infty} 1$
  preferred when exact solns complicated
- (ch3) generating function is a formal power series (x is not treated as a number; powers of x are placeholders for sequence terms) that encodes sequence $\langle a \rangle$ by using the terms as coefficients
  eg ordinary generating function is $\sum_{n=0}^{\infty} a_n x^n$
  note: manipulating expressions for generating function can lead to explicit or asymptotic formula for a_n
- finite sum
  eg (§4.1) inclusion-exclusion principle technique that alternately overcounts and undercounts the desired set until each desired element is counted exactly once; commonly uses finite sum

1.1. CLASSICAL MODELS.

Remark. This section
- several elementary techniques as "principles"
- several elementary models used in solving more difficult counting problems

Definition. Elementary Principles
- sum principle ("counting by cases")
  if a finite set A is partitioned (split into disjoint nonempty sets whose union is A) into $B_1, ..., B_k$ then $|A| = \sum_{i=1}^{k} |B_i|$
  note: key step is recognizing what the index of summation means in describing the pieces
- product principle ("counting by stages")
  If elements of A are built via successive choices where the *number of options* (not the actual options) for each choice is indep of outcomes of earlier choices
  then $|A|$ is the product of the number of options for successive choices
  eg: $|A \times B| = |A|\,|B|$
- principle of counting two ways
  when two formulas count the same set, then their values are equal
  note: must devise a set they both count
  Hermann Weyl: this property is one of deepest in math
- bijection principle
  if there is a bijection ("1-1 correspondence") ($f : A \to B$ st $\forall b \in B\ \exists! a \in A$ st $f(a) = b$) from one set to another
  then they have same size
  note: this principle is just the def of equal size
  note: bijection maps parts of problem into model (previously solved problem)
  note: need cononical sets of known sizes
  note: can prove bijections directly or inductively
- pigeonhole principle
  the max (min) in a set of numbers is at least as large (small) as the avg
  in particluar, placing more than kn objects in k boxes puts more than k objects in some box
  note: for a precise pf, consider the contrapositive
  note: will generalize in ch 10
- polynomial principle
  If two polynomials are equal for all positive integer values of all the variables
  then they are the same polynomial
  in particular, they have the same coefs
  note: can generalize to more than one variable by induction (ex32-34,eg1.1.18)
  note: for one variable, just need equality at degree+1 points; but will need infinitely many points to prove equality for all positive integers for all variables

Remark.
- first two principles break the problem into smaller parts
  organize the set to simplify counting it
- second two principles are usually requested in "combinatorial proof"
- will invoke above principles in arguments

Example.
- number of ways to pair (pairs are unordered, no repetition) $2n$ people
- show $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$
- ("interchange order of sum") show $\sum_{i=1}^{m}\sum_{j=1}^{n} f(i,j) = \sum_{j=1}^{n}\sum_{i=1}^{m} f(i,j)$
- within the set $[n] := \{1, ..., n\}$, the number of k-element subsets equals the number of $(n-k)$-element subsets
- the number of $0,1$-lists of length n equals the number of subsets of $[n]$
- let $f : A \to B$ and $g : B \to A$ are injections
  case A, B finite, then $|A| = |B|$ and f, g are bijections
  case A, B infinite, then A, B have same cardinality, but f, g need not be bijections (ex 47)

Remark. (application to probability)
for finite "sample space" U of "equal probability" outcomes
the probability of event $A \subset U$ is $P(A) = \frac{|A|}{|U|}$
probability of a property is probability of all events with that property
will use probability to motivate problems
ch 14 for more sophisticated use of probability in counting problems

WORDS, SETS, AND MULTISETS

Definition.
- An alphabet is a set [I will try to use $[n] := \{1, ..., n\}$ for any finite alphabet]
- A k-word ("n-ary k-tuple") is a list of k elements from an alphabet $[n]$
- A simple word has destinct letters
- A k-set is a k-element set
- A k-set in S is a k-element subset of S
- A multiset from S is a selection from S which allows repetitions
  intuition: equivalence classes of k-words from n which use same numers of letters

Definition. let $n \in \mathbb{N}_0$
- n-factorial $n! = \prod_{i=0}^{n-1}(n-i)$
- falling factorial $n_{(k)} = \prod_{i=0}^{k-1}(n-i)$
- rising factorial $n^{(k)} = \prod_{i=0}^{k-1}(n+i)$
- binomial coef $\binom{n}{k} := \begin{cases} \frac{n_{(k)}}{k!} & 0 \le k \le n; n, k \in \mathbb{N}_0 \\ 0 & \text{ow} \end{cases}$

where convention: operation on empty set is identity ie $0! = n_{(0)} = n^{(0)} = 1$

Remark. Ways to have k items from alphabet $[n]$

	no repetiton	repetition allowed
ordered ("arrangement")	simple k-word ("k-permutation") $n_{(k)}$	k-words n^k
unordered ("selection")	subset of size k ("k-combinations") $\binom{n}{k}$	multisets of size k $\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$

Remark. classical models:
- (k-words)
  - build word in stages
  - k-words from $[n]$ corresponds to function $f : [k] \to [n]$, where $f(i) = i$th element of word
- (simple words)
  - build word in stages
  - special case: simple word of length n from set $[n]$ is permutation (in word form, other forms later) of $[n]$
- (subsets, k-sets)

- subset (of any size) $A \subseteq [n]$ corresponds to incidence vector $a_1...a_n$ st $a_i = \begin{cases} 0 & : i \notin A \\ 1 & : i \in A \end{cases}$
  (analogously, functions $f : [n] \to \{0,1\}$)
  specifically for k-sets, word has k ones
- (multisets)
  - Each k-element multiset corresponds to a nonnegative integer vector [histogram] $x_1, ..., x_n$ of multiplicities
    so model with words of k dots and $n-1$ bars
  - (integer-sum problem) number of k-element multisets from $[n]$ equals nonnegative integer solns to $\sum_{i=1}^n x_i = k$
  - note: this problem can pop up with sizes and types reversed

Remark. bijections between above table cells (may repeat parts of above rmk)

- (k-words and simple words)
  simple words are a special case of k-words
- (k-sets and binary words)
  binary words with k ones, $n-k$ zeros
- (multisets and binary words)
  multiplicity vectors of multiset correspond to words of bars and dots
- (multisets and compositions)
  tweak the integer-sum model or the bars,dots model
- (simple k-words and k-sets)
  divide or multiply by the number of ways to order

Proposition.

- (binomial thm) for $n \in \mathbb{N}_0$, $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$
- there are $\binom{k-1}{n-1}$ compositions (special multisets where each part (multiplicity of each element) a positive integer) of k with n-parts

1.2. IDENTITIES.

Remark.

- want identities (standard formulas) to evaluate sums involving binomial coefs
- types of arguments for proving identities:
  - combinatorial - provide deep understaning and strong results
  - inductive - involve tedious manipulation
  - algebraic - easy to manipulate a known identity like binomial coefs

  will focus on combinatorial arguments first

LATTICE PATHS AND PASCAL'S TRIANGLE

Remark. will model k-sets in $[n]$ with lattice paths

Definition.

- A lattice point is a vector with integer coordinates
- A lattice step changes one coordinate by one
- A lattice walk from a lattice point (usually origin) moves by lattice steps
- A lattice path is a lattice walk where each step *increases* one coordinate

Remark. model:
(lattice path and binary k-words)
binary n-word with k ones corresponds to lattice path from $(0,0)$ to $(k, n-k)$ (ie length n, k horizontal steps)

[ELEMENTARY IDENTITIES]

Remark. ntn: $\sum_k$ is over all nonzero terms

Theorem. *(Elementary identities)*

1) $\binom{n}{k} = \binom{n}{n-k}$ *("complementation identity")*
generalization: §1.3
2) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ *("Pascal's formula")*
generalization: §1.3
3) $k\binom{n}{k} = n\binom{n-1}{k-1}$ *("committee-chair identity")*
generalization:
$\binom{k}{l}\binom{n}{k} = \binom{n}{l}\binom{n-l}{k-l}$ *("subcommittee identity")*
4) $\sum_k \binom{n}{k} = 2^n$
generalization:
$\sum_k r^k \binom{n}{k} = (r+1)^n$
5) $\sum_{k=0}^n \binom{k}{r} = \binom{n+1}{r+1}$ *("summation identity")*
generalization:
$\sum_{k=-m}^n \binom{m+k}{r}\binom{n-k}{s} = \binom{m+n+1}{r+s+1}$
6) $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$
generalization:
$\sum_k \binom{m}{k}\binom{n}{r-k} = \binom{n+m}{r}$ *("Vandermonde's Convolution")*

Remark.

- Pascal's triangle can be built from Pascal's formula
- identity (1) is *not* a polynomial in n since requires $n \in \mathbb{N}_0$
  identity (2) extends as a polynomial (ex6)
  identities (5),(6) are polynomials in n, m
- next, will further extend identity (4) to a polynomial
  but first, extend binomial coef as a polynomial

Definition.
recall $\binom{n}{k}$ in terms of $n_{(k)} := \prod_{i=0}^{k-1}(n-i)$
The extended binomial coefficient is
$\binom{u}{k} := \begin{cases} \frac{1}{k!}\prod_{i=0}^{k-1}(u-i) & \text{for } u \in \mathbb{R}, k \in \mathbb{N}_0 \\ 0 & ow \end{cases}$
note: it is a polynomial in u of degree k
note: case $u < 0$: $\binom{u}{k} = $ "$\frac{u_{(k)}}{k!}$" $= (-1)^k\binom{-u+k-1}{k}$

Theorem. *(extended binomial thm)*
$(1+x)^u = \sum_{k \ge 0} \binom{u}{k} x^k$ *where* $u, x \in \mathbb{R}$, $|x| < 1$
note: this extends identity (4general)

Remark. methods of evaluating sums

- induction - requires knowing the value of the sum in advance
  eg ex13 pascal's idenity
- bijection proof
  eg ex10-41,eg1.3.45
- use identities already proved - might need to manipulate a sum into form of identity

Example. (application) for any polynomial $p(x)$, find $\sum_{i=1}^n p(i)$

DELANNOY NUMBERS [AND TAXI BALLS]

Remark. will generalize lattice paths with delannoy paths
will count taxi balls size
will find model delannoy numbers with taxi ball size

Definition.

- the delannoy number $d_{m,n}$ is the number of paths from $(0,0)$ to (m,n) st each step is in $\{(1,0),(0,1),(1,1)\}$
- the central delannoy numbers are $d_{n,n}$
  eg $d_{2,2} = 13$
- (geometrically) a taxi ball of radius m in n dimensions is the set of lattice points in $\mathbb{Z}^n$ that are within m unit coordinate steps from one point (usually the origin)

Remark. models (found throughout section)

- a taxi ball (radius m in $\mathbb{Z}^n$) and the set of all integer n-tuples st the absolute value of the entries sum to at most m, (? integer-sum model, ? incidence vector)
- model lattice paths with dots and bars
- model compositions of $m+1$ with $n+1$ parts with special delannoy walks
- size of taxi ball and delannoy number

Proposition.

- $d_{m,n} = \sum_k \binom{m}{k}\binom{m+n-k}{m} = \sum_k \binom{m}{k}\binom{n+k}{m}$
- the size of the taxi ball of radius m in $\mathbb{Z}^n$ is $\sum_k \binom{n}{k}\binom{m}{k}2^k$
- (delannoy (1889)) $d_{m,n}$ is the size of the taxi ball of radius m in $\mathbb{Z}^n$

Remark.
symmetric in m,n: def of $d_{m.n}$; formula for size of taxi ball: $\sum_k \binom{n}{k}\binom{m}{k}2^k$
not symmetric: formula $d_{m.n} = \sum_k \binom{m}{k}\binom{m+n-k}{m} = \sum_k \binom{m}{k}\binom{n+k}{m}$

1.3. APPLICATIONS.

Remark. previous sections: proving given identities, finding sets to model given formulas
this section: counting a given set

GRAPHS AND TREES

Definition.

- edge is pair of vertices
- graph is determinded by set of edges
- tree is connected graph without cycles
  intuition: built by iteratively adding new leaves, so $n-1$ edges
- functional digraph of $f : S \to S$ wrt vertex set S is a graph having an edge from each $x \in S$ to $f(x)$
  intuition: each vertex is the tail of one edge, so there are $|S|$ edges

Remark. models

- Egecioglu-Remmel bijection between all functional digraphs $f : [n] \to [n]$ and tree with vertex set $[n]$

Proposition.

- for vertex set $[n]$, there are $2^{\binom{n}{2}}$ graphs
- (Cayley's thm) for a vetex set $[n]$, there are n^{n-2} trees

MULTINOMIAL COEFFICIENTS

Remark. will generalize binomial coefs with multinomial coefs
will use to count trees with specific counts of edges entering each vertex

Definition. multinomial coeficients $\binom{m}{k_1,...,k_n}$ count m-words in $[n]$ with multiplicities $k_1, ..., k_n$ of each letter ie $\sum k_i = m$
ntn: $P(k; k_1, ..., k_n)$

Remark. models

- trees with vertex set $[n]$ st vertex $1, ..., n$ has degrees $d_1, ..., d_n$
  and $(n-2)$-words from $[n]$ with $d_i - 1$ copies of i for each i
  hint: use functional digraph

Proposition.

- $\binom{m}{k_1,...,k_n} = \frac{m!}{\prod_{i=1}^n k_i!}$
- permuting $k_1, ..., k_n$ does not change value
- $P(k_1+k_2; k_1, k_2) = C(k_1+k_2, k_1)$
- the number of trees with vertex set $[n]$ in which vertices $1, ..., n$ have degrees $d_1, ..., d_n$ is $\frac{(n-2)!}{\prod(d_i-1)!}$
- (multinomial thm) $\left(\sum_{i=1}^n x_i\right)^k = \sum \binom{k}{k_1,...,k_n}\prod_{i=1}^n x_i^{k_i}$ where $n \in \mathbb{N}_0$ and rhs sum is over nonegative integer n-tuples $(k_1, ..., k_n)$ with sum k
- extend Pascal's formula: $\binom{m}{k_1,...,k_n} = \binom{m-1}{k_i-1,k_2...,k_n} + ... + \binom{n-1}{k_1,...,k_{n-1},k_n-1}$
- (application)(Fermat's Little thm) if $n \in \mathbb{Z}$ and p prime, then $n^p \equiv n (\text{mod } p)$
  furthermore, $n^{p-1} \equiv 1 (\text{mod } p)$ when $p \nmid n$ ("p does not divide n")

THE BALLOT PROBLEM

Remark. Bertrand's ballot problem (from combinatorial probability):
Let candidates A, B receive a, b votes with $a \ge b$ when votes are counted in random order, what is the probability that A never trails

Remark. models

- $(a+b)$-words with a As and b Bs st each initial segment has at least as many As
  and lattice paths to (a,b) that don't cross diagonal line

Proposition. Among the lists with a,b copies of A,B,
there are $\binom{a+b}{b}-\binom{a+b}{a+1}$ st every initial segment has at least as many As as Bs

Remark.
- answer to motiviating question:
  Probability(good list)$=\frac{\binom{a+b}{b}-\binom{a+b}{a+1}}{\binom{a+b}{a}} = 1 - \frac{b}{a+1} = \frac{a-b+1}{a+1}$
- can generalize to strict inequality $a > b$
- next, an identity for central binomial coefs $\binom{2m}{m}$
  but first a lem

Lemma. *$\binom{2m}{m}$ counts the following types of lattice paths of length $2m$ starting at $(0,0)$*
a) those ending in (m,m)
b) those never rising above line $y = x$
c) those never returning to line $y = x$

Theorem. *for $n \in \mathbb{N}_0$, $\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$*

CATALAN NUMBERS

Remark.
- Catalan numbers arise in many counting problems, and their formulas can be found in many ways
- will revisit Catalan numbers: derivation §2.1,2, computations of $\sum_{i=1}^{n} i^2$, delannoy numbers and the derangement problem, and enumerating various types of permutations and partitions

Definition.
- Catalan numbers are $C_n := \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$
- generalized Catalan numbers are $\frac{1}{qn+1}\binom{(q+1)n}{n}$
  case $q = 1$: Catalan numbers
  case $q = 2$: Fuss-Catalan numbers
- ballot path of length $2n$ is a lattice path to (n,n) that never rises above $y = x$
- ballot list of length n is a $0,1$ n-word st initial segment has at least as many 1s as 0s
- q-ballot list is list of n 1s and qn 0s st every initial segment has at least q times as many 1s as 0s

Remark. models counted by catalan numbers
- ballot paths and ballot lists of length 2n
- binary trees of $n+1$ leaves (grouping $n+1$ terms)
- orderd trees with n edges
- triangulations of a conves $(n+2)$-gon
- noncrossing pairings of $2n$ points on a circle
- noncrossing partitions of $[n]$
- stack-sortable permutations of $[n]$

Proposition.
- the nth Catalan number counts both ballot paths and ballot lists
- if p,q are relatively prime ($\gcd(p,q) = 1$)
  then the number of lattice paths to (p,q) that do not rise above line $py = qx$ is $\frac{1}{p+q}\binom{p+q}{p}$
- for $q \in \mathbb{N}$, the number of lattice paths to (n, qn) that do not rise above line $y = qx$ is the generalized Catalan number
- generalized catalan numbers count q-ballot lists

[TREES]

Definition.
- rooted graph is a graph with one vertex distinguished as root
- tree with root r has parent of vertex v is the neighbor of v on path to r
  note: root has no parent
- children are its other neighbors
- a leaf in a rooted tree is a vertex having no children
  note: root is not a leaf unless it is the only vertex
- ordered tree ("rooted plane tree") is a rooted tree in which the children of each vertex are given a fixed (left-to-right) linear order
- binary tree is an ordered tree in which every vertex has zero or two children, but in some contexts can have one child
- the left (right) subtree of a binary tree is rooted at the left (right) child of the root

Remark. models
- binary trees and parenthizations (groupings are binary operations among leaves)
  eg $((a,b),c),d$ where a,b,c,d are leaves
- binary trees with $n+1$ leaves and ballot lists

Proposition.
- a binary tree with $n+1$ leaves has n non-leaf vertices and $2n$ edges
- the number of binary trees with $n+1$ leaves is C_n
- there are C_n triangulations of a convex $(n+2)$-gon

2. RECURRENCE RELATIONS

Remark. will first focus on one (integer) parameter counting problems

Definition.
- counting sequence $\rangle a \langle := \langle a_0, a_1, ...\rangle$ is the sequence of solns to a one (integer) parameter counting problem
- recurrence relation ("recurrence") expresses nth value of counting sequence in terms of earlier values and a function of n
  $a_n = g(n, a_0, a_1, ..., a_n)$

- order ("degree") k recurrence means $a_n = g(n, a_n, ..., a_{n-k})$
  linear recurrence is $a_n = g_1(n)a_{n-1} + ... + g_k(n)a_{n-k} + f(n)$ where $g_i(n), f(n)$ are indep of $\langle a\rangle$
  homogeneous recurrence means $f(n) = 0$, inhomogeneous otherwise
- ("recursive") soln of recurrence eqn is an exact (or asymptotic) formula for the values in the counting sequence

Remark. steps to solve counting problem recursively:
- use combinatorial arguments to obtain recurrence eqn
- obtain k initial values for order k recurrence
- find recursive soln one of the following ways:
  - somehow obtain soln formula (see §2.1), check formula against initial values (base cases), verify by induction that the formula is "valid" for larger n
  - characteristic eqn method for special recurrences
  - generating function method for more general cases
  - substitution techniques
  - special methods for asymptotic solns

Remark. Operator interpretation:
alternative form of recurrence: $g(a_0, ..., a_n) = 0$
so want to find null-space of operator g
But will not use this interpretation

Example. Count a_n :=# of list from $\{0,1\}$ of length n
soln: every list of length n arises by extending a length $n-1$ list with 0 or 1.
so $a_n = 2a_{n-1}$ for $n \geq 1$.
note: $a_0 = 1$ ie there is one way to do nothing.
so $a_n = 2^n$ by induction
takeaway: sometimes solns can be built from smaller problems of same type

2.1. OBTAINING RECURRENCES.

3. GENERATING FUNCTIONS

Remark.
- this section: systematic methods to solve many counting problems
- types of generating functions:
  ordinary generating function (OGF) - useful in studying multisets and selections and in partitions of integers
  exponential generating function (EGF) - useful for enumerating "labeled" structures

3.1. ORDINARY GENERATING FUNCTIONS.

Remark. will assume one parameter n (any others fixed) as the index of sequence $\langle a\rangle$, and seek soln a_n

Definition.
- A formal power series is expression of the form $\sum_{n=0}^{\infty} a_n x^n$
- The OGF ("ordinary enumerator") for sequence $\langle a\rangle$ is the formal power series $\sum_{n=0}^{\infty} a_n x^n$

Remark.
- "generating functions for $\langle a\rangle$" refers to any expressions whose formal power series expansion is $\sum_{n=0}^{\infty} a_n x^n$
- x is a "formal variable" (not numerical; placeholders for term a_n)
  so "function" $A(x) := \sum_{n=0}^{\infty} a_n x^n$ "generates" the sequence of coefs a_n
- convergence of formal power series is irrelevant but convergence can be helpful in extracting asymptotoic formulas for coefs (§3.4)

Example.
- Let a_n be the number of binary lists of length n.
  generating function: $A(x) := \sum_{k=0}^{\infty} 2^n x^n$
  $A(x)$ the enumerator of binary lists, indexed by length
- Let $a_{n,k}$ be the number of k-subsets of an n-set for fixed n
  generating function: $A_n(x) := \sum_{k=0}^{n}\binom{n}{k}x^k$
  $A_n(x)$ is the enumerator of subsets of $[n]$ indexed by size
  another generating function is $(1+x)^n$ since binomial thm
- for more than one parameter, must choose which parameter is index of summation
  might be easier to form a generating function in two variables $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k}x^n y^k$

MODELLING COUNTING PROBLEMS

Remark. the sum and product principles are modelled by the sum and product of formal power series so can model a sequence of counting problems

Definition.
- the sum and product of formal power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are
  $\sum_{n_0}^{\infty}(a_n + b_n)x^n$ and $\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_j b_{n-j}\right)x^n$
- The convolution of sequences $\langle a\rangle$,$\langle b\rangle$ is the sequence of coefficients $\sum_{j=0}^{n} a_j b_{n-j}$
- recall: operator $[x^k]$ applied to power series returns coef of x^k
  ntn: $[x^k]A(x) = a_n$ ie no parentheses around argument $A(x)$

Proposition.
- Let a_k, b_k, c_k count elements with index k in sets A,B,C
  The OGF for $\langle c\rangle$ is the product of OGFs for $\langle a\rangle$ and $\langle b\rangle$

iff for all k, the elements in C with index k correspond to the ordered pairs (α, β) st $\alpha \in A, \beta \in B$, and k is the sum of the indices of α and β
pf: this correspondence is precisely $c_k = \sum_{j=0}^{k} a_j b_{k-j}$
note: this generalizes to objects built in n stages

- two power series are equal iff their coefs are equal
- $(1 + x + x^2 + ...)^n = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k$
pf: both are generating functions for the number of multisets from $[n]$
lhs from examples in this chapter
rhs binomial coef from ch 1.

Example. multisets

- intuition: consider all 6-multisets from $\{a, b, c\}$. the multiset with multiplicities 3,1,2 corresponds to $x^3x^1x^2 = x^6$
- number of k-multisets from [1]:
note: there is one way to have any k-multiset from [1]
generating function: $\sum_{k=0}^{\infty} x^k$ indexed by size
- number of k-multisets from [2]:
intuition: k coins on a table, each head or tails
generating function 1: $\sum_{k=0}^{\infty}(k+1)x^k$
pf: the number of 1s is 0 to k
generating function 2: $\left(\sum_{k=0}^{\infty} x^k\right)^2$
pf: the first contributes j and the second contributes (n-j)
- the number of k-multisets from [4]
intuition: k coins on table, two types of coin, each heads or tails
generating function 1: $\sum_{k \geq 0} \left[\sum_{i=0}^{k}(i+1)(k-i+1)\right] x^k$ indexed by number of coins
pf: there are i of the first coin and $k-i$ of the second
generating funtion 2: $\left[\sum_{k=0}^{\infty}(k+1)x^k\right]^2$ indexed by number of coins
pf: (?)
- number of multisets from $[n]$
generating function1: $\sum_{k \geq 0} a_k x^k$ by size where a_k is the number of k-multisets from $[n]$
pf: sum principle
generating function2: $\left(\sum_{k=0}^{\infty} x_k\right)^n$
pf: use the prop generalized to n objects

[VECTOR SPACES, ALGEBRAS, INVERSES]

Remark.

- the definition of sum makes the set of formal power series in x an infinite dimensional vector space
- the definition of product makes it an algebra, with identity $x^0 + 0x^1 + 0x^2 + ...(= 1)$
- note: multiplication of power series is commutative

Definition. define $A(x)^{-1} = B(x)$ iff $A(x)B(x) = 1$

Proposition.

- $A(x)$ has a multiplicative inverse iff $[x^0]A(x) \neq 0$
pf: $B(x) = \frac{C(x)}{A(x)}$ so $A(x)$ must be nonzero
- Consider generating function $(1-x)^{-1}$ for $n \in \mathbb{N}$
the formal power series expansion is $(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k$
pf: $(1-x)$ is the inverse of $\sum_{k=0}^{\infty} x^k$
so $(1-x)^n \left(\sum_{j=0}^{\infty} x^j\right)^n = 1$

Example.

- number of multisets from [1] with multiplicities restricted to set S
generating function: $\sum_{k \in S} x^k$
- number of multisets from $[n]$ with multiplicities restricted to set S
generating function: $\left(\sum_{k \in S} x^k\right)^n$
- number of ordinary subsets from [n] (ie multisets with restriction $S = \{0, 1\}$)
generating function: $(1+x)^n$
- compositions from $[n]$ (ie multisets with $S = \{1, 2, ...\}$)
generating function: $\left(\sum_{k=1}^{\infty} x^k\right)^n$ which equals $\left(\frac{x}{1-x}\right)^n$
so $[x^k]\left(\frac{x}{1-x}\right)^n = [x^{k-n}](1-x)^{-n} = \binom{k-n+n-1}{n-1} = \binom{k-1}{n-1}$
- number of multisets from $[n]$ st only even multiplicites
generating function $1 + x^2 + x^4 + ...$ which equals $(1-x^2)^{-1}$
- number of 20-multisets from $\{a, b, c\}$ with at least three bs and at most four cs
$[x^{20}] \frac{1}{1-x} \frac{x^3}{1-x} \frac{1-x^5}{1-x} = [x^{20}] \frac{x^3 - x^8}{(1-x)^3} = [x^{17}] \sum_k \binom{k+3-1}{3-1} x^k - [x^{12}] \sum_k \binom{k+3-1}{3-1} x^k = \binom{19}{2} - \binom{14}{2}$

Remark.

- ntn: it is neater not to truncate sum, even if we only need up to certain power
- "combinatorial argument to obtain generating function" means using using generating functions to model the sum (case) and product (stage) choices when building objects being enumerated. additivity of size (the index parameter) modeled by accumulating multiplcities in corresponding power of formal variable
- generalize to m parameters
the coef of $x_1^{k_1} \cdots x_m^{k_m}$ is the number of objects with parameters $k_1, ..., k_m$.

Example.

- (ex17) flipping nickels and dimes (?)
- number of k-subsets of $[n]$
note: $a_{n,k} = \binom{n}{k}$
generating function: $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} x^k y^n = \sum_{n=0}^{\infty} (1+x)^n y^n = \frac{1}{1-y-xy}$
note: $(1-y-xy) \sum \sum \binom{n}{k} x^k y^n = 1$ is read as $a_{n,k} - a_{n-1,k} - a_{n-1,k-1} = 0$, ie Pascals formula. So can use generating function to obtain recurrence

PERMUTATION STATISTICS

Remark.

- permutation statistic - parameter for counting properties of permutations eg number of fixed points, number of cycles
- Let $\mathfrak{S}_n$ be the set of permutations of $[n]$
note: $|\mathfrak{S}_n| = n!$
note: the permutations of $[n]$ form the symmetric group on n elements under the composition operation, so $\mathfrak{S}_n$ stands for "symmetric"
- we focus on counting, but permutation statistics have applications in algebraic combinatorics and in the theory of special functions

Definition.

- an inversion is pair (σ_i, σ_j), $\sigma \in \mathfrak{S}_n$, st $i < j$ and $\sigma_i > \sigma_j$
- notations for a permutations
  - the 2-line form ("tabular representation")
  eg $\binom{123456789}{461752398}$
  - the word form
  eg 461752398
  - the cycle represntation
  eg (4731)(62)(89)(5)
  - the canonical cycle represntation puts least element of each cycle first, and puts cycles in decreasing order of least element
  eg (89)(5)(26)(1473)

Proposition.

- The enumerator of $\mathfrak{S}_n$ by the number of inversions is $(1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1})$
pf: (?)
check: plug $x = 1$ to get $n!$
- if $f(\sigma)$ denotes the canonical cycle representations of a permutation σ written without parentheses,
then f is a bijection from $\mathfrak{S}_n$ to itself
pf: surjection since for each permutation, insert parentheses before each left-to-right mimumum
bijection since surjection of finite set to itself
note: this bijection similar idea to Egecioglu-Remmel
- Let $c(n, k)$ be the number of elements in $\mathfrak{S}_n$ with k cycles
The enumerator of $\mathfrak{S}_n$ by the number of cycles is $C_n(x) := \sum_{k=1}^{n} c(n, k) x^k = x^{(n)}$
pf1: recall $x^{(n)} := \prod_{i=1}^{n}(x + i - 1)$
(?)
check: plug $x = 1$ to get $n!$
pf2: using polynomial principle
(?)

[EULERIAN NUMBERS]

Definition.

- for $\sigma \in \mathfrak{S}_n$, a run is a maximal increasing string in the word form
- descent (ascent) at i means $\sigma_i > \sigma_{i+1}$ ($\sigma_i < \sigma_{i+1}$)
- the descent set $D(\sigma)$ is the subset of $[n-1]$ at which descents occur
- the Eulerian number $A(n, k)$ is the number of permutations of $[n]$ having k runs

Example.

- permutation 791368452 has four runs (79,1368,45,2) and three descents (at 2,6,8), and five ascents
- $A(4, 2) = A(4, 3) = 11$

Proposition.

- every permutation of $[n]$ with k runs has $k-1$ descents and $n-k$ ascents; the ascent set of σ is $[n-1] - D(\sigma)$
- $A(n, k) = A(n, n+1-k)$
pf: reversing the word form (?)
- (Worpitzky's identity (1883))
$A(n, k)$ satisfies $x^n = \sum_{k=1}^{n} A(n, k) \binom{x-k+1}{n}$
pf: barred permutations (?)
- $A(n, k) := \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n$
pf1: invert Worpitzky's identity (?)
pf2: (ex38) substitute the original relation into the desired expression, and the unwanted contribution vanishes
pf3: (ex37) induction

Remark. the nth Eulerian polynomial is the generating function $A_n(x) := A(n, k)x^k$
note: proof of most recent thm used $A_n(x) = (1-x)^{n+1} \sum_{k \geq 1} k^n x^k$ (which involves an infinite sum)

3.2. COEFFICIENTS AND APPLICATIONS.

Remark.

- a generating function's formal power series can be manipulated like power series, with the additional freedom of ignoring convergence (ie dont care if zero radius of convergence)
- operations on formal power series are defined to agree with operations on corresponding power series that converge at numerical values
- so can prove things using power series or formal power series
- evaluation of $A(x)$ at numerical value x is allowed when the corresponding sum converges
eg $A(0) := a_0$
- derivative of formal power series $\sum_{k=0}^{\infty} a_n x^n$ is the formal power series $\sum_{k=0}^{\infty} n a_n x^{n-1}$
note: termwise differentiable agrees with taylor series
ntn: $A'(x)$

OPERATIONS AND SUMMATIONS

Example.

- generating function $(1-x)^{-n}$ has formal power series $\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}x^k$
  pf1: differentiate n times both sides of $\frac{1}{1-x} = \sum_{k\ge 0} x^k$
  pf2: product of formal power series
- find generating function for negative binomial expansion $\sum_{k\ge 0}\binom{k}{r}x^k$
  soln: $\sum_{k\ge 0}\binom{k}{r}x^k = \sum_{k\ge r}\binom{k}{r}x^k = \sum_{k\ge 0}\binom{k+r}{r}x^{k+r} = \frac{x^r}{(1-x)^{r+1}}$

Remark. given: generating function, find: coefs a_n. methods:

- $a_n = \frac{1}{n!}A^{(n)}(0)$
  but repeatedly differentiating $A(x)$ an may be difficult or infeasible
- start with known power series eg for $(1+x)^n$, $(1-x)^{-n}$, $\frac{1-x^{n+1}}{1-x}$, e^x
  operate on the function and power series simultaneously, maintaining equality, until function looks like given generating function
  use the following properties of these manipulations:
  1) $c_n = a_n + b_n$
     $\Leftrightarrow C(x) = A(x) + B(x)$
  2) $c_n = \sum_{i=0}^n a_i b_{n-i}$ for all n
     $\Leftrightarrow C(x) = A(x)B(x)$
  3) $b_n = \begin{cases} a_{n-r} & \text{for } n \ge r \\ 0 & n < r \end{cases}$
     $\Leftrightarrow B(x) = x^r A(x)$
     intuition: shift indices
  4) $b_n = na_n$
     $\Leftrightarrow B(x) = xA'(x)$
  5) $c_n = \sum_{i=0}^n a_i$ for all n
     $\Leftrightarrow C(x) = \frac{A(x)}{1-x}$ (special case of 2)
  6) a) $b_n = \begin{cases} a_n & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$
     $\Leftrightarrow B(x) = \frac{1}{2}[A(x) + A(-x)]$
     b) $b_n = \begin{cases} a_n & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$
     $\Leftrightarrow B(x) = \frac{1}{2}[A(x) - A(-x)]$
     intuition: cancel out odd or even terms
  7) $b_n = \begin{cases} a_{n/m} & m \mid n \\ 0 & m \nmid n \end{cases}$
     $\Leftrightarrow B(x) = A(x^m)$
     intuition: substituting by $y = x^m$ spreads out the terms

Example. manipulations to find coefs

- (number of chaired committees from n people)
  start with $\sum\binom{n}{k}x^k = (1+x)^n$
  differentiate and set $x = 1$
  $\sum_k \binom{n}{k} = n2^{n-1}$
  both sides count chaired committees from n people
- number of even subsets of $[n]$
  start with $\sum\binom{n}{k}x^k = (1+x)^n$
  $\sum\binom{n}{2i}x^{2i} \overset{(6)}{=} \frac{1}{2}\left[(1+x)^n + (1-x)^n\right]$
  set $x = 1$
  $\sum\binom{n}{2i} = 2^{n-1}$
  both sides count the even subsets of $[n]$
- Given generating function $A(x)$ indexed by whatever
  setting $x = 1$ gives sum of all coefs
  setting $x = -1$ gives (sum of even coefs)-(sum of odd coefs)
  - generating function $(1+x)^n$ enumerates subsets of $[n]$ by size
    $x = 1$ yields all 2^n subsets
    $x = -1$ yields 0, so there are equal number for even and odd
  - generating function $\prod_{j=0}^{n-1}\sum_{i=0}^{j} x^i$ enumerates permutations of $[n]$ by number of inversions
    $x = 1$ yields all $n!$ permutations
    $x = -1$ yields 0, so there are equal number for even and odd
  - generating function $\prod_{j=1}^{n}(x+j-1)$ enumerates permutations of $[n]$ by number of number of cycles
    $x = 1$ yields all $n!$ permutations
    $x = -1$ yields 0, so there are equal number for even and odd
- recall $\sum_{k=0}^n p(k)$ for polynomial p
  [how is this related to below?]
  find identity for $\sum_{k=0}^n k^2$
  start with $\sum_{k\ge 0} x^k = \frac{1}{1-x}$
  $\sum_{k\ge 0} kx^k \overset{(4)}{=} \frac{x}{(1-x)^2}$, $\sum_{k\ge 0} k^2x^k \overset{(4)}{=} \frac{x+x^2}{(1-x)^3}$
  then $\sum_{k=0}^n k^2 \overset{(5)}{=} [x^n]\frac{x+x^2}{(1-x)^4} = [x^{n-1}]\frac{1}{(1-x)^4} + [x^{n-2}]\frac{1}{(1-x)^4} = \binom{n-1+3}{3} + \binom{n-2+3}{3} = \frac{(n+1)n(2n+1)}{6}$
- prove Vandermonde convolution identity
  $\sum_{k=0}^r \binom{m}{k}\binom{n}{r-k} = [x^r]\left(\sum_k \binom{m}{k}x^k\right)\left(\sum_k\binom{n}{k}x^k\right) = [x^r](1+x)^m(1+x)^n = [x^r](1+x)^{m+n} = \binom{m+n}{r}$
- find identity for $\sum_{k=0}^n k(n-k)$
  note this is convolution with $a_k = b_k = k$
  $\sum_{k=0}^n k(n-k) = [x^n]\left(\sum kx^k\right)\left(\sum kx^k\right) = [x^n]\left(\frac{x}{(1-x)^2}\right)^2 = [x^n]\frac{x^2}{(1-x)^4} = [x^n]\sum_{k=0}^{\infty}\binom{k+4-1}{4-1}x^{k+2} = \binom{n+1}{3}$
- (central binomial coefs)
  show $\sum_{k=0}^n \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$ (thm1.3.14)
  lhs is a convolution with $a_k = b_k = \binom{2k}{k}$
  observe $\sum_{k\ge 0}\binom{2k}{k}x^k = \frac{\mathrm{d}}{\mathrm{d}x}xC(x)$ where $C(x) = \sum_{k\ge 0}\frac{1}{k+1}\binom{2k}{k}x^k = \frac{1-\sqrt{1-4x}}{2x}$ is the Catalan generating function
  So $\frac{\mathrm{d}}{\mathrm{d}x}x\frac{1-\sqrt{1-4x}}{2x} = \frac{1}{\sqrt{1-4x}}$
  so $\left(\sum_{k\ge 0}\binom{2k}{k}x^k\right)^2 = \frac{1}{1-4x}$
  so $[x^n]\frac{1}{1-4x} = 4^n$

SNAKE OIL

Remark. the Snake Oil method evaluates sums by interchanging the order of summation

Example.

- evaluate $\sum_{k\ge 0}\binom{k}{n-k}$
  let $a_n := \sum_{k\ge 0}\binom{k}{n-k}$
  $\sum_{n\ge 0} a_n x^n = \sum_{k\ge 0} x^k \sum_{n\ge 0}\binom{k}{n-k}x^{n-k} = \sum_{k\ge 0} x^k(1+x)^k = \frac{1}{1-x-x^2}$
  note: last expression enumerstes 1, 2-words by their sum, so this is an OGF for the adjusted Fibonacci numbers, so $a_n = \hat{F}_n$
- show $\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{n}{k}\binom{m}{k}2^k$ (thm1.2.13 relating $D_{m,b}$ and taxi ball)
  multiply both sides by x^n, sum over n
  will show both sides have same generating function
  note: will use $\sum_{r\ge 0}\binom{r}{k}x^r = \frac{x^k}{(1-x)^{k+1}}$
  lhs: $\sum_k \binom{m}{k}x^{-k}\sum_{n\ge 0}\binom{n+k}{m}x^{n+k} = \frac{x^m}{(1-x)^{m+1}}\sum_k\binom{m}{k}x^{-k} = \frac{x^m}{(1-x)^{m+1}}(1 + x^{-1})m = \frac{(1+x)^m}{(1-x)^{m+1}}$
  note: $\binom{m}{k} = 0$ unless $k \le m$ so all the terms needed to form $(1-x)^{m+1}$ were present
  rhs: $\sum_k \binom{m}{k}2^k\sum_{n\ge 0}\binom{n}{k}x^n = \frac{1}{1-x}\sum_k\binom{m}{k}2^k\left(\frac{1}{1-k}\right)^k = \frac{1}{1-x}\left(1 + \frac{2x}{1-x}\right)^m = \frac{(1+x)^m}{(1-x)^{m+1}}$
- show $\sum_{k=m}^n c(n,k)\binom{k}{m} = c(n+1, m+1)$
  intution: $c(n,k)$ is the number of permutations of $[n]$ with k cycles
  $\sum_{m\ge 0} x^m \sum_{k\ge 0} c(n,k)\binom{k}{m} = \sum_k c(n,k)\sum_m\binom{k}{m}x^m = \sum_k c(n,k)(1+x)^k \overset{thm3.1.19}{=} (1+x)^{(n)}$
  next extract coef of x^m using trick
  $\sum_{k=m}^n c(n,k)\binom{k}{m} = [x^m](1+x)^{(n)} = [x^{m+1}]x^{(n+1)} = c(n+1, m+1)$
- optional - introduce free variable when parameter appears more than once

3.3. EXPONENTIAL GENERATING FUNCTIONS.

Remark.

- words are the ordered analogue of multisets
  ie choose k-multiset from $[n]$, but choose elements in order
- words, and more generally labeled structures, have analogous generating function to multisets
- will model compound problems as product of these generating functions

MODELING LABELED STRUCTURES

Definition.

- the exponential generating function (EGF) ("exponential enumerator") for a sequence $\langle a\rangle$ is $\sum a_n\frac{x^n}{n!}$ or any "function" $A(x)$ with this formal power series
- the binomial convolution of $\langle a\rangle$ and $\langle b\rangle$ is $\langle c\rangle$ st $c_n = \sum_{j=0}^n\binom{n}{j}a_jb_{n-j}$ for $n \in \mathbb{N}_0$
- A labeled structure is formed by using finitely many distinct labels each once
  eg words are structures with labels as positions
- symmetric family A_S of labeled structures with label set S means the number of structures depends only on $|S|$
  ie the number of structures with a given label set depends only on the number number of labels
  note: $a_{|S|} = |A_S|$
  eg number of k-words from fixed $[n]$ is n^k, so symmetric

Proposition.

- the following are equivalent
  1) The EGF for $\langle c\rangle$ is the product of EGFs for $\langle a\rangle$ and $\langle b\rangle$
  2) $\langle c\rangle$ is the binomial convolution of $\langle a\rangle$ and $\langle b\rangle$
  3) for symmetric families A, B, C and label set S, the objects in C_S correspond to triples (T, α, β) st $T \subset S$, $\alpha \in A_T$, $\beta \in B_{S-T}$
  pf: (1⇔2) $\left(\sum_{n=0}^{\infty} a_n\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^n \frac{a_j}{j!}\frac{b_{n-j}}{(n-j)!}\right)x^n = \sum_{n=0}^{\infty}\left(\sum_{j=0}^n\binom{n}{j}a_jb\right.$
  (1⇔3) (?)
- above generalizes to products of n factors
  pf: induction, see ex13

Remark.

- EGFs have algebraic properties like OGFs
- the index of the EGF is the number of labels
  eg length of word
- eg for words, each label (position) must be used once, but letters can be used any number of times
  so for k-words from $[n]$, must allocate k labels into n sets (some of which may be empty)
  the EGF is the product of the EGFs associated with each of the n sets
- when allocating labels to subproblems, it doesnt matter which labels go where, only how many
- by prop, multiplying EGFs useful iff product of component problems yields binomial convolution iff symmetric families
- multiplying EGFs corresponds to forming labeled structures in stages
- binomial convolution introduces factor $\binom{n}{k}$ that counts allocations of labels to subproblems

Example.

- number of k-words from [1]
  note: there is one k-word from [1]
  EGF: $\sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x$ indexed by length
- number of k-words from $[n]$
  note: there are n^k k-words from $[n]$

EGF: $\sum_{k=0}^{\infty} n^k \frac{x^k}{k!} = e^{nx}$ indexed by length
could also use product $(e^x)^n = e^{nx}$

- number of k-words from [26] eg Roman alphabet
  method1: there are 21 consonants and 5 vowels
  so there are $\sum_{j=0}^{k} \binom{k}{j} 21^j 5^{n-j}$ k-words since first choose how many consonants, then fill those in, then fill in the rest with vowels
  binomial thm gives $(21+5)^k = 26^k$ k-words
  method2: above is binomial convolution
  so $e^{21x}e^{5x} = e^{26x}$ so 26^k k-words
- (example without words)
  number of ways to put n distinct flags (labels) on r flagpoles, where order of flags on each pole is important
  - case $r = 1$:
    note there are $n!$ structures with label set $[n]$
    so EGF is $\sum_n n! \frac{x^n}{n!} = \sum_n x^n = (1-x)^{-1}$
    indexed by number of flags
  - case $r = 2$
    choose number of flags on first pole, then arrange those, then arrange rest on other pole
    so EGF is $\sum_n \left(\sum_{k=0}^{n} \binom{n}{k} k!(n-k)!\right) \frac{x^n}{n!} = (1-x)^{-2}$ indexed by number of flags
  - case r arbitrary:
    EGF is $(1-x)^{-r}$ indexed by number of flags
    note: this is also EGF for multisets from $[r]$
    rewrite EGF: $(1-x)^{-r} = \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} x^n = \sum_{n=0}^{\infty} (n+r-1)_{(n)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} r^{(n)} \frac{x^n}{n!}$ (just like ex1.1.34)
- (words with restricted use of letters)
  - unrestricted multiplicity:
    factor when building words: e^x
  - must use a particular letter:
    factor when building words: $e^x - 1$
    intuition: cant have zero-length word using that letter
  - a letter used at most once (ie simple words)
    factor when building words: $1+x$ (note: $\frac{x}{1!} = x$)
- number of simple words from $[n]$
  EGF $(1+x)^n$ indexed by length
  pf: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} n_{(k)} \frac{x^k}{k!}$
- Fix alphabet $[n]$. compare OGFs indexed by size with EGFs indexed by length

multiplicity	multisets (OGF)	words (EGF)
unrestricted	$(1-x)^{-n}$	e^{nx}
≤ 1 of each type	$(1+x)^n$	$(1+x)^n$
≥ 1 of each type	$\left(\frac{x}{1-x}\right)^n$	$(e^x-1)^n$

- extract coef when EGF is sum of powers of e^x
  - words with particular letter used an even number of times
    factor when building words is $\frac{1}{2}(e^x - e^{-x})$
    coef of $\frac{x^k}{k!}$ is $\frac{1}{2}(1^k - (-1)^k)$
    similarly, for odd multiplicity, $\frac{1}{2}(e^x + e^{-x})$
  - ternery words with even number of 0s, odd number of 1s, any number of 2s
    EGF $\frac{1}{2}(e^x + e^{-x})\frac{1}{2}(e^x - e^{-x})e^x = \frac{1}{4}(e^{3x} - e^{-x})$
    the coef of $\frac{x^k}{k!}$ is $\frac{1}{4}(3^k - (-1)^k)$
  - 0,1-lists with even multiplicities of both 0 and 1
    EGF $\frac{1}{4}(e^x + e^{-x})^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$
    coef of $\frac{x^k}{k!}$ is $2^{k-2} + (-2)^{k-2}$, but adjusted by $\frac{1}{2}$ for $k = 0$ since const term $\frac{2e^0}{4}$

[STIRLING NUMBERS]

Remark. next application for enumerating words where each letter must be used

Definition.

- recall partition of set A is a set of disjoint nonempty subsets with union A
- blocks are the subsets of a partition
- ordered partition when the blocks have distinct names
- Stirling number of the second kind $S(n.k)$ is the number of partitions of $[n]$ into k nonempty blocks
- signless Stirnling number $c(n,k)$ is the number of permutations of $[n]$ with k cycles
- Stirling number of the first kind $s(n,k) = (-1)^{n-k} c(n,k)$

Proposition.

- $S(n.k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$
  pf: will count the number of ordered partitions, $k!S(n,k)$, then divide by $k!$
  EGF $\sum_{n=0}^{\infty} k!S(n,k) \frac{x^n}{n!}$
  assigning elements of $[n]$ into k blocks is like assigning a label from $[k]$ to each element in $[n]$
  this is modeled by n-words from $[k]$
  blocks are nonempty so each letter in $[k]$ must be used
  so another EGF: $(e^x - 1)^k$
  binomial thm gives another EGF: $\sum_{i=0}^{k} (-1)^i \binom{k}{i} e^{x(k-i)}$
  expand to $\sum_{i=0}^{k} (-1)^k \binom{k}{i} \sum_{n \ge 0} (k-i)^n \frac{x^n}{n!}$
  interchange order of summation to get desired sum as coef of $\frac{x^n}{n!}$, modulo dividing by $k!$ as discussed
  pf2: §4.1 using inclusion-exclusion principle
  pf3: ex25 using Eulerian number
- for $n \in \mathbb{N}_0$,
  (1) $\sum_{k=0}^{n} S(n,k) x_{(k)} = x^n$
  (2) $\sum_{k=0}^{n} s(n,k) x^k = x_{(n)}$
  intuition: Stirnling numbers transform between two bases for the vector space of polynomials
  pf: (1) sufficient to prove for $x \in \mathbb{N}$ since polynomial principle
  count the number of n-words from $[x]$
  rhs: product principle
  lhs: let index k be number of letters used
  note: $S(n,k)$ ways to partition the n positions into k blocks
  note: $x_{(k)}$ ways to assign blocks to distinct letters in $[x]$
  (2) $x_{(n)} = (-1)^n (-x)^{(n)} = (-1)^n \sum_{k=0}^{n} c(n,k)(-x)^k = \sum_{k=0}^{n} (-1)^{n+k} c(n,k) x^k = \sum_{k=0}^{n} s(n,k) x^k$
- $\sum_{k \ge 0} S(n,k) s(k,n) = \delta_{n,m}$ for $n, m \in \mathbb{N}_0$
  intuition: The two types of Stirling numbers form inverse matrices
  pf: $x^n = \sum_{k=0}^{n} S(n,k) x_{(k)} = \sum_{k=0}^{n} S(n,k) \sum_{m=0}^{k} s(k,m) x^m = \sum_{m=0}^{n} \left[\sum_{k=m}^{n} S(n,k) s(k,m)\right] x^m$
  coefs of x^n must be 1, all others must be 0

Example.

- for $m = 0$, $S(0,k) = s(0,k) = \delta_{0,k}$ where Kronecker delta function $\delta_{i,j} := \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$
- thus, application1.2.7, can be written $x^k = \sum_{j=0}^{k} S(k,j) x_{(j)} = \sum_{j=0}^{k} j! S(k,j) \binom{x}{j}$

EGF ANALOGUES OF OGF APPLICATIONS

Proposition.

- (binomial inversion formula)
  For $\langle a \rangle$, $\langle b \rangle$
  $a_n = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}$ iff $b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{n-k}$
  pf: let EGFs $A(x), B(x)$ for $\langle a \rangle$, $\langle b \rangle$
  multiply both equalities by $\frac{x^n}{n!}$ and sum over n
  then statement becomes $A(x) = e^x B(x)$ iff $B(x) = e^{-x} A(x)$

Example.

- evaluate $\sum_{k=0}^{n} \binom{n}{k} m^k$
  method1: binomial thm
  method2: let $a_k = m^k$, $b_{n-k} = 1$
  EGFs: e^{mx} and e^x
  so $\sum_{k=0}^{n} \binom{n}{k} m^k = [\frac{x^n}{n!}] e^{mx} e^x = [\frac{x^n}{n!}] e^{(m+1)x} = (m+1)^n$
- more generally, to evaluate sum of the form $\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$
  find $EGFs$ for $\langle a \rangle$ and $\langle b \rangle$ and extract coef of the product
- for derangements numbers, an EGF is $D(x) = \frac{e^{-x}}{1-x}$ and a formula is $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  pf: recall (ex2.1.6) $n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}$
  Binomial inversion thm $\Rightarrow$ $D_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  recurrence rhs has EGF $\frac{1}{1-x}$
  recurrence rhs is binomial convolution of $a_k = 1$ and $b_{n-k} = D_{n-k}$
  so $\frac{1}{1-x} = e^x D(x)$
  method2: take recurrence $n! = \sum \binom{n}{k} D_{n-k}$, multiply by $\frac{x^n}{n!}$, sum over n, giving EGF
  find D_n since multiplying by $\frac{1}{1-x}$ sums first n terms of power series
- recall (ex2.1.6) $D_n = (n-1)(D_{n-1} + D_{n-2})$, $D_0 = 1$, $D_1 = 0$
  note: there should be an EGF since pemutations are labeled structures
  multiply by (trick:) $\frac{x^{n-1}}{(n-1)!}$ and sum over n
  $\sum_{n \ge 2} D_n \frac{x^{n-1}}{(n-1)!} = \sum_{n \ge 2} D_{n-1} \frac{x^{n-2}}{(n-1)!} + \sum_{n \ge 2} D_{n-2} \frac{x^{n-1}}{(n-2)!} =$
  Shift factorials and powers to agree with subscripts
  see book, involves diff eqs
  $D(x) = \frac{e^{-x}}{1-x}$

Remark. see book for optional story of Bernoulli numbers B_n

THE EXPONENTIAL FORMULA

Remark.

- will generalize ideas of this section
- will build "general" structures from "component" structures
  eg sets from subsets, graphs from connected subgraphs
- will relate "general" structure's EGF $G(x)$ to its "component" structure's EGF $C(x)$
- the following def and rmk dont seem relevant, so just skip to prop

Definition.

- sequence $A_1(x), A_2(x), \ldots$ of formal power series converges to formal power series $\sum_{k=0}^{\infty} a_k x^k$ means for every k exists n_k st $n \ge n_k$ implies $[x^k] A_n(x) = a_k$
- truncation of a formal power series $C(x) = \sum_{k=0}^{\infty} c_k x^k$ is $C_n(x) := \sum_{k=0}^{n} c_k x^k$
- composition of formal power series $A(B(x)) := \sum_{k \ge 0} a_k (B(x))^k$
- composition converges means $\lim_{n \to \infty} A_n(B_n(x))$ converges

Remark.

- When does a composition converge?
  1) if $B(0) = 0$, then $[x^k] A_n(B_n(x))$ is fixed for $n \ge k$, so composition converges
  2) if $B(x)$ is a polynomial, then $[x^k] A_n(B_n(x))$ is fixed for n large, so composition converges
- If $A(B(x))$ is itself a formal power series, then it equals $\lim_{n \to \infty} A_n(B_n(x))$

Proposition.

- (The exponential formula)
  let EGF $G(x)$ for symmetric family of "general" labeled structures
  let EGF $C(x)$ for symmetric family of "component" labeled structures
  let $G(0) = 1$ and $C(0) = 0$
  If general structures are formed by partitioning a

set of labels and placing a component structure on the label of each block (?)
then $G(x) = e^{C(x)}$
note: every component structure has at least one element, and the one general structure with no elements has no components
pf: take general structure with label set $[n]$
partition the labels into blocks $B_1, ..., B_k$
build component structures with each block
EGF $C(x)$, indexed by number of labels, enumerates number of components from each block
divide by $k!$ to cancel indexing the blocks
The EGF for the general structure with k components is $\frac{C(x)^k}{k!}$
the general structure is the sum over number of components, $G(x) = \sum_{k \geq 0} \frac{C(x)^k}{k!} = e^{C(x)}$

- Let $C(x) = \sum c_n \frac{x^n}{n!}$ and $G(x) = \sum g_n \frac{x^n}{n!}$
If $G(x) = e^{C(x)}$ then $c_n = g_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} c_k g_{n-k}$ for $n \geq 1$
intuition: given $G(x) = e^{C(x)}$, can compute coefficients in $C(x)$ recursively using coefs of $G(x)$
pf: let general structure with label set $[n]$
label n belongs in a component of size $k \geq 1$
complete the structure by choosing the remaining $k-1$ labels of this component, choosing a component structure on these k labels, and choosing the general structure on the remaining $n-k$ labels
thus $g_n = \sum_{k=1}^{n} \binom{n-1}{k-1} c_k g_{n-k}$
the term for $k = n$ counts n-element components

Example.
- number of graphs using connected subgraphs as components
  - aside: one EGF is $G(x) = \sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}$, indexed by number of vertices, since there are $2^{\binom{n}{2}}$ graphs with vertex set $[n]$
  - Consider graphs with n vertices
    Find EGF for number of ways to have k connected subgraphs
    case $k = 2$: $\sum_{j=0}^{n} \binom{n}{j} c_j c_{n-j}$, where $c_0 = 0$
    So EGF $\sum_{n \geq 0} \left(\frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} c_j c_{n-j} \right) \frac{x^n}{n!} = \frac{C(x)^2}{2}$, where divided by 2 since either connected subgraph can be considered the "first" one
    case k general: EGF is $\frac{C(x)^k}{k!}$
  - relate $C(x)$ to $G(x)$
    can enumerate number of n-vertex graphs with EGF $\sum_{k=0}^{\infty} \frac{C(x)^k}{k!} = e^{C(x)}$ indexed by number of connected subgraphs
    so $G(x) = e^{C(x)}$
    intuition: $G(x)$ is composition of two formal power series
  - find c_n, ie the number of connected graphs from n vertices
    $c_n = 2^{\binom{n}{2}} - \sum_{k=1}^{n-1} \binom{n-1}{k-1} 2^{\binom{n-k}{2}} c_k$
- number of partitions of an n-set using subsets as components
  - EGF $C(x) = e^x - 1$ indexed by size, since there is one way to make a given subset
  - by exponential law, $G(x) = e^{e^x - 1}$
  - aside: Bell number (not to be confused with Bernoulli number "B_n") $B_n = \sum_{k=0}^{n} S(n,k)$ is the number of partitions of $[n]$, so EGF $e^{e^x - 1} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} S(n,k) \frac{x^n}{n!}$
- number of permutations of $[n]$, using cycles as components
  - there are $(j-1)!$ cycles from j labels
    so EGF $C(x) = \sum_{j \geq 1} \frac{x^j}{j} = -ln(1-x)$
  - by exponential formula, EGF $G(x) = e^{-ln(1-x)} = \frac{1}{1-x}$, indexed by size n
    so there are $n!$ permutations of $[n]$
- number of involutions (permutations whose square is identity) from $[n]$ using cycles of length 1 or 2 as components
  - $C(x) = x + \frac{x^2}{2}$ since involutions have cycles of length one or two
  - $G(x) = e^{x + \frac{x^2}{2}}$, indexed by length

Remark. see book for optional remark on generalizing exponential formula to compositional formula

THE LAGRANGE INVERSION FORMULA (OPTIONAL)

3.4. PARTITIONS OF INTEGERS.

Definition.
- a partition of integer n is a multiset of positive integers with sum n
  ntn: canonically write them in nonincreasing order, since order unimportant
- a composition of n is a partition with order
- parts of a partition or composition are the numbers, can be repeated

Example. a bridge hand is a partition of 13 into at most four parts
eg distribution 5440 has 5 of one suit, 4 of each of two suits, and none of the other suit

GENERATING FUNCTION METHODS

Remark.
- want to enumerate partitions with specific properties, indexed by their sum
- OGF is appropriate because we are partitioning identical units
- one way to specify a partition is by the number of times each integer is used as a part
  so will build the generating function with a factor for each number allowed to be a part

Definition.
- partition number $p(n)$:=number of partitions of n
- $a_{n,k}$ is the number of partitions of n using parts in $[k]$

Proposition.
- a) $a_{n,k}$ has OGF $\prod_{i=1}^{k} \frac{1}{1-x^i}$
  pf: want OGF to be indexed by sum n
  model for each n is nonnegative integer soln to $1e_1 + 2e_2 + ... + ke_k = n$
  the options for $i \in [k]$ is encoded by OGF $1 + x^i + x^{2i} + ... = \frac{1}{1-x^i}$
  eg x^{2i} corresponds to two copies of i
- $p(n)$ has OGF $\prod_{i=1}^{\infty} \frac{1}{1-x^i}$
  pf: infinite product allows all positive integer parts

Example.
- restrictions
  - partitions of $[n]$ with largest part k have OGF $x^k \prod_{i=1}^{k} \frac{1}{1-x^i}$
    pf: ensure at least one k is used by removing x^0
  - number of partitions of n in which all parts are odd, 1 is used at least three times, and 3 is used at most three times is $x^2(1-x)^{-1}(1+x^3+x^6+x^9)\frac{1}{1-x^5}\frac{1}{1-x^7}\cdots$
- $a_{n,1} = 1$ and $a_{n,2} = \lfloor \frac{n}{2} \rfloor$
- the number of partitions of n using parts in $\{1,2,3\}$ is the nearest integer to $\frac{1}{12}(n+3)^2$
  pf: OGF: $\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}$
  rewrite denominator linear factors (?): $(1-x)^3(1+x)(1-\omega x)(1-\omega^2 x)$ where $\omega = e^{\frac{2\pi i}{3}}$
  expand OGF using partial fractions (ex2):
  $\prod_{i=1}^{3} \frac{1}{1-x^i} = \frac{1/6}{(1-x)^3} + \frac{1/4}{(1-x)^2} + \frac{17/72}{1-x} + \frac{1/8}{1+x} + \frac{1/9}{1-\omega x} + \frac{1/9}{1-\omega^2 x}$
  apply negative bonomial expansion:
  $a_{n,3} = \frac{1}{6}\binom{n+2}{2} + \frac{1}{4}(n+1) + \frac{17}{72} + \frac{(-1)^n}{8} + \frac{1}{9}(\omega^n + \omega^{2n})$
  note: $\frac{1}{6}\binom{n+2}{2} + \frac{1}{4}(n+1) = \frac{1}{12}(n+3)^2 - \frac{1}{3}$
  move $\frac{1}{12}(n+3)^2$ to lhs, apply triangle inequality
  $\left| a_{n,3} - \frac{1}{12}(n+3)^2 \right| \leq \frac{7}{72} + \frac{1}{8} + \frac{2}{9} < \frac{1}{2}$

[ASYMPTOTIC BEHAVIOR]

Remark.
- for asymptotic analysis, treat generating function as a power series and worry about convergence
- reasoning prop for $a_{n,3}$ can be used for any fixed k
  the dominant contribution comes from $(1-x)^{-k}$
- will find asymptotic formula for $a_{n,k}$ for fixed k ie as $n \to \infty$
- will find upper and lower bounds for asymptotic behavior of $p(n)$
  (lower bound given as a remark)
- will state asymptotic behavior of $p(n)$, but wont prove since would need methods from analytic number theory

Proposition.
- the number of partitions of n using parts in $[k]$ is asymptotic to $\frac{n^{k-1}}{k!(k-1)!}$
  pf: (?)
- number of partitions of n using parts in $[\sqrt{n}]$ (ie k grows slowly enough with n) has lower bound $\frac{n^{\sqrt{n}}}{\left(\frac{\sqrt{n}}{e}\right)^{2\sqrt{n}}} = e^{2\sqrt{n}}$
  intuition: not many partitions can have very large parts
  intuition2: partition of n with largest part k corresponds to partitions of $n-k$ with parts no bigger than k, and $p(n-k)$ is smaller than $p(n)$
  pf: use Stirling's appx on result for fixed k
- $p(n+1) < \frac{\pi}{\sqrt{6n}} e^{\pi \sqrt{\frac{2}{3} n}}$ which is asymptotically about $e^{2.565\sqrt{n}}$
  pf: use OGF $P(x) = \sum_{n \geq 0} p(n) x^n = \prod_{k \geq 1} (1 - x^k)^{-1}$
  see book for obtaining a numerical bound on $lnP(x)$ and then on $lnp(n+1)$
- $p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$
  pf: Hardy-Ramanujan [1918], not given

Remark. see optional remark about using complex analysis, analytic functions, residue, Cauchy integral formula,

FERRERS DIAGRAMS

Remark.
- a geometric approach to partition will give combinatorial proofs of many identities
- partition argument: to show number of partitions is equinumerous, can find bijection between partitions
  iff OGFs are equal

Definition.
- The Ferrer's diagram of a partition λ of n is an array of n dots with λ_i(:=the ith largest part) dots in row i
  note: rows are left justified, and each at least as long as the row below it
- the conjucate λ^* of partition λ has transposed Ferrers diagram of λ
  eg see book for diagram of partition 5,3,1,1 to 4,2,2,1,1

Proposition.
- The number of partitions of n with largest part k equals the number of partitions of n into k parts
  pf: the conjugation bijection converts every partition with k parts into a partition with largest part k

- partitions with parts in $[k]$ correspond to partitions with at most k parts
  pf: similar
- $\prod_{i=1}^{\infty}(1+x^i)=\prod_{i=1}^{\infty}(1-x^{2i-1})^{-1}$
  pf: lhs is OGF for partitions into distinct parts
  rhs is OGF for partitions into odd parts
  by next prop, they are equal
  see ex12
- The number of partitions of n into distinct parts equals the number of partitions of n into odd parts
  pf: let function f take partition μ with odd parts to a partition λ with distinct parts by iteratively combining two identical parts until no identical parts remain
  invert by iteratively breaking each even part into two parts
- The OGF for congruence classes of integer triangles (integer-length sides), indexed by perimeter (ie congruent iff their side lengths form the same partition of the perimeter), is $\frac{x^3}{(1-x^2)(1-x^3)(1-x^4)}$
  pf1: partition perimeter m into parts $a \geq b \geq c$ st triangle inequality $a < b + c$
  this partition corresponds to partitions with parts in $\{2,3,4\}$ with at least one 3, since see book about reassembling Ferrers diagram
  pf2: ex 15, develops OGF for these congruence classes, does not require prior knowledge of the answer
- (Euler's Identity)
  $\prod_{i=1}^{\infty}(1+x)^{2i-1}=1+\sum_{k=1}^{\infty}\frac{x^{k^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2k})}$
  pf1: Euler used algebra
  pf2: both expressions enumerate self-conjugate partitions (those uncahnged by transposing the diagram)
  see book for picture and Durfee square

BULGARIAN SOLITAIRE (optional)
DISTRIBUTION MODELS (summary)

Remark.
- many classical counting problems can be modeled as distributions of objects into boxes, with conditions on the objects, the boxes, and the allowed distributions
- distributions of k objects into n boxes
  can require one object per box (injective I), use of all boxes (surjective S), or arbitrary (unrestricted U) distributions
  can require {identical, distinct} objects in {distinct,indistinguishable ("identical")} boxes
  this gives 12 possibilities which can be put into a 3x4 table

	k Distinct Objects n Distinct Boxes	k Identical Objects n Distinct Boxes
U	k-word from $[n]$ $\overline{P}_{n,k}=n^k$ $\overline{P}_{n,k}=n\overline{P}_{n,k-1}$ $\sum_{k\geq 0}\overline{P}_{n,k}x^k=\frac{1}{1-nx}$ $\sum_{k\geq 0}\overline{P}_{n,k}\frac{x^k}{k!}=e^{nx}$ $\overline{P}_{n,k}=\overline{P}_{n-1,k}+\sum_i\binom{n}{i}\overline{n-1}k-i$	k-multisets from $[n]$ $\overline{C}_{n,k}=\binom{k+n-1}{n-1}=(-1)^k\binom{-n}{k}$ $\sum_{k\geq 0}\overline{C}_{n.k}x^k=\left(\frac{1}{1-x}\right)^n$ $\overline{C}_{n.k}=\overline{C}_{n-1,k}+\overline{C}_{n,k-1}$
I $\leq$ 1	simple k-word from $[n]$ $P_{n,k}=\frac{n!}{(n-k)!}$ $\sum_{k\geq 0}P_{n,k}\frac{x^k}{k!}=(1+x)^n$ $P_{n,k}=P_{n-1,k}+kP_{n-1,k-1}$ $P_{n,k}=nP_{n-1,k-1}$	k-sets from $[n]$ $C_{n,k}=\binom{n}{k}=\frac{n!}{k!(n-k)!}$ $\sum_{k\geq 0}C_{n,k}x^k=(1+x)^n$ $C_{n,k}=C_{n-1,k}+C_{n-1,k-1}$
S $\geq$ 1	partitions of k objects into n labeled blocks $\overline{S}_{k,n}=n!S_{k,n}$ $\overline{S}_{k,n}=\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^k$ $\sum_{k\geq 0}\overline{S}_{k,n}\frac{x^k}{k!}=(e^x-1)^n$ $\overline{S}_{k,n}=n(\overline{S}_{k-1,n-1}+\overline{S}_{k-1,n})$	compositions of integer k into n positive parts solns to $\sum_{i=1}^{n}e_i=k$, $e_i\geq 1$ $q_{n,k}=\binom{k-1}{n-1}$ $\sum_{k\geq 0}q_{n,k}x^k=\left(\frac{x}{1-x}\right)^n$ $q_{n,k}=q_{n,k-1}+q_{n-1,k-1}$
	k Distinct Objects n Indistinguishable Boxes (group objects without labels on groups)	k Identical Objects n Indistinguishable Boxes (group a set of k identical dots)
U	partitions of k-set into at most n blocks $\sum_{i=0}^{n}S_{k,i}$ For $n=k$, is the Bell number B_k $\sum_{k\geq 0}B_k\frac{x^k}{k!}=e^{e^x-1}$ $B_k=\sum_{i=0}^{k-1}\binom{k-1}{i}B_i$	Partitions of integer k into at most n parts $\overline{p}_{k,n}=\sum_{i=0}^{n}p_{k,i}$ $\sum_{k\geq 0}\overline{p}_{k,n}x^k=\frac{1}{\prod_{i=0}^{n}(1-x^i)}$ $\overline{p}_{k,n}=\overline{p}_{k-n,n}+\overline{p}_{k,n-1}$ $\overline{p}_{k,n}=\overline{p}_{k-n,n}+\overline{p}_{k,n-1}$
I $\leq$ 1	1 way if $k\leq n$ 0 ways if $k>n$	1 way if $k\leq n$ 0 ways if $k>n$
S $\geq$ 1	partitions of k-set into n blocks $S_{k,n}=\frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^k$ $\sum_{k\geq 0}S_{k,n}\frac{x^k}{k!}=\frac{1}{n!}(e^x-1)^n$ $\sum_{k\geq 0}S_{k,n}x^k=\frac{x^n}{\prod_{i=1}^{n}(1-ix)}$ $S_{k,n}=S_{k-1,n-1}+nS_{k-1,n}$	Partitions of integer k into n parts $\sum_{k\geq 0}p_{k,n}x^k=\frac{x^n}{\prod_{i=1}^{n}(1-x^i)}$ $p_{k,n}=p_{k-n,n}+p_{k-1,n-1}$

- an 18-possibility table also considers the order of distinct elements inside a box
  eg arrangements of k flags on n flagpoles is in cell UDD with additional care for order of objects within the boxes
  eg $c(k,n)$(= number of permutations of $[k]$ with n cycles) is in cell SDI and then cyclically arranging within the box
- a 30-possibility table allows arbitrary number of boxes
  includes Bell numbers and the total number of partitions of an integer
  includes multiplicities within boxes, introducing multinomial coefs
- when constraints involve more than one box, then distribution models are not useful
  eg permutations of $[k]$ with n runs is surjective, but the largest object in one box must exceed the smallest object in the next

4. FURTHER TOPICS

Remark. this chapter exhibits indirect counting techniques
- the inclusion-exclusion principle - combinatorial explanation for many alternating sums
- Polya-Redfield method - count equivalence classes of objects made indistinguishable by symmetry operators
- Young tableaux - from theory of representations of symmetric groups; connections to permutations and sorting

4.1. THE PRINCIPLE OF INCLUSION-EXCLUSION (PIE).

Remark.
- PIE is the most common sieve method (counting method allowing a smaller desired set to survive a process of overcounting and undercounting)
- [Brualdi - PIE generalizes summation principle, since can count sets which overlap]

THE BASIC PRINCIPLE

Definition.
- set U is the universe
- Let $A_1,...,A_n\subseteq U$
  Let $T\subseteq[n]$ ie index set
  define $f(T):=|\{x: x\in A_i \text{ iff } i\in T\}|$
  ie the number of elements in A_i for $i\in T$ AND not in A_i for $i\notin T$
  and $f(\emptyset)=\left|\cap_{i=1}^{n}\overline{A}_i\right|=\left|\overline{\cup_{i=1}^{n}A_i}\right|$ is the elements outside of all A_i
  intuition: "cells" of Venn diagram
  ntn: f(12) insead of $f(\{1,2\})$

Proposition.
- $|\cap_{i\in S}A_i|=\sum_{T\supseteq S}f(T)$
  pf: see Venn diagram for intuition of intersection and sum of cells
- (Principle of Inclusion-Exclusion - PIE)
  Let $A_1,...,A_n\subseteq U$, $T\subseteq[n]$, and f defined above
  The inclusion-exclusion formula is
  $f(T)=\sum_{S\supseteq T}(-1)^{|S|-|T|}\left|\cap_{i\in S}A_i\right|$
  pf: will show each $x\in U$ contributes equally to both sides
  rhs: note: there is a term for each $S\supseteq T$
  case $x\in\cap_{i\in S}A_i$: x contributes $+1$ (-1) for $|S|-|T|$ even (odd)
  case $x\notin\cap_{i\in T}A_i$: x has no contribution since $\cap_{i\in S}A_i\subseteq\cap_{i\in T}A_i$
  define $R(x):=\{i\in[n]: x\in A_i\}$ as the indices of sets containing x
  note: $x\in\cap_{i\in S}A_i$ iff $T\subseteq S\subseteq R(x)$
  note: each S corresponds to a subset of $R(x)-T$
  so for each subset of $R(x)-T$, x contributes $+1$ or -1 corresponding to whether $|S|-|T|$ is even or odd
  note: nonempty sets have equal numbers of even and odd subsets
  case $R(x)\neq T$: x contributes net 0, since cancel out
  case $R(x)=T$: x contributes 1 and nothing to cancel it
  lhs: $T=R(x)$ so exactly those x contribute
  pf2: count the number of elements in restricted universe $U':=\cap_{i\in T}A_i$ outside of $A'_i:=A_i\cap U'$ whenever $i\notin T$
  (draw picture for intuition)
  $f'(\emptyset)=\sum_{S'\subseteq[n]-T}(-1)^{|S'|}\left|\cap_{i\in S'}A'_i\right|$
  term of this formula equal those in inclusion-exclusion formula (?)

Remark.

- the inclusion-exclusion formula *inverts* identity $|\cap_{i\in S}A_i| = \sum_{T\supseteq S} f(T)$ st we can compute $f(T)$ using values of $|\cap_{i\in S}A_i|$ for various S
  like inversion principles in ch3 for sequences $\langle a\rangle$, $\langle b\rangle$, but indexed by subsets of $[n]$ instead of by $\mathbb{N}_0$
- applications often ask for $f(\emptyset)$ or $|U| - f(\emptyset)$
  model: let $A_i \subseteq U$ be the elements violating the ith constraint, then the elements meeting all constraints is $f(\emptyset)$
  the formula is $f(\emptyset) = \sum_{S\subseteq[n]}(-1)^{|S|}\,|\cap_{i\in S}A_i|$
  intution: start with $|\cap_{i\in\emptyset}A_i| = |U|$ and rest is alternating sum of intersections
- tricks
  - above formulas have $|\cap A_i|$ and $f(T)$, choose whichever is easier to compute
  - demorgan: $\overline{\cup_{i=1}^n A_i} = \cap_{i=1}^n \overline{A_i}$
    transform to union or intersections, whichever is easier to compute
  - symbol sets of same size all contribute with the same sign, so can group them
    $f(\emptyset) = \sum_{k=0}^n (-1)^k h_k$ where $h_k = \sum_{S:|S|=k} |\cap_{i\in S}A_i|$
    but beware: h_k does not equal the number of elements lying in at least k of the sets (?).
    The sum overcounts elements lying in more than k sets (?)
  - in some applications, $|\cap_{i\in S}A_i|$ depends only on $|S|$
    then the sets of size k make the same contribution $g_k = |\cap_{i\in S}A_i|$ for $|S| = k$
    then inclusion-exclusion formula is
    $f(\emptyset) = \sum_{k=0}^n (-1)^k \binom{n}{k} g_k$
- when asked to evaluate: $\sum_{k=0}^n (-1)^k\binom{n}{k}c_k$
  model as inclusion-exclusion computation for number of elements outside $A_1, ..., A_n$ in U
  this model requires $|U| = c_0$ and $|A_i| = c_1$, and $c_k = |\cap_{i\in S}A_i|$ for $|S| = k$; creativity may be needed to find natural sets with these sizes
  similar to evaluating convolutions using generating functions
  dont need to know in advance what we are counting

Example.

- the classical model - counting unions using intersections
  - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
    intuition: first two terms double count elements in both sets
  - $|A_1 \cup A_1 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$
    intuition: first three terms overcount each element in multiple sets, the next three terms cancel elements in two sets, which cancels elements in all three sets so need the last term
  - above two examples are equal to $|U| - |\overline{A_1 \cup A_2}|$ and $|U| - |\overline{A_1 \cup A_2 \cup A_3}|$
- counting using $|\cap_{i\in S}A_i| = \sum_{T\supseteq S} f(T)$
  let $A_1, A_2, A_3 \subseteq U$ (see book for venn diagram)
  - $|A_1| = f(1) + f(12) + f(13) + f(123)$
  - $|A_1 \cap A_2| = f(12) + f(123)$
- (counting derangements of $[n]$)
  let $U = \mathfrak{S}_n$
  let A_i be the set of permutations of $[n]$ that fix element i
  then $D_n = |\cap_{i=1}^n \overline{A_i}|$ which is $f(\emptyset)$
  note: number of permutations fixing S is $|\cap_{i\in S}A_i| = (n - |S|)!$ since after fixing those in S, can permute rest arbitarily, which depends only on $|S|$
  (book leaves following conclusion to subsection "restricted permutations")
  so $D_n = \sum_{k=1}^n (-1)^k\binom{n}{k}(n-k)! = n!\sum_{k=0}^n \frac{(-1)^k}{k!}$
- (application to number theory)
  Euler's totient $\phi(m)$:=count of numbers in $[m]$ which are relatively prime to m
  $\phi(m) = \sum_{S\subseteq[n]}(-1)^{|S|}\frac{m}{\prod_{i\in S}p_i}$ where $p_1, ..., p_n$ are the distinct prime factors of m
  pf: see book
- (multisets with restricted usage)
  - count number of m-multisets with n types of objects (eg 20-multisets of red, white, and blue marbles)
    soln: multiset formula $\binom{m+n-1}{n-1}$
  - find: $a_{m,n,r}$:=number of m-multisets with n types with fewer than r of each type
    soln: aside: model: integer-equation model with integer solns to $x_1 + ... + x_n = m$ st $0 \le x_i < r$
    let U be the set of solns for the unrestricted case, so $|U| = \binom{m+n-1}{n-1}$
    let $A_i \subset U$ be the multisets where type i violates the limit, ie where $x_i \ge r$
    Then $|A_i| = \binom{m-r+n-1}{n-1}$ since know $x_i \ge r$
    If k constraints are violated, then kr units are known and $m - kr$ are distributed arbitrarily
    thus $|\cap_{i\in S}A_i| = \binom{m-|S|r+n-1}{n-1}$, which depends only on $|S|$
    inclusion-exclusion formula: $a_{m,n,r} = \sum_{k=0}^n (-1)^k\binom{n}{k}\binom{m-kr+n-1}{n-1}$
  - special case: $m = n(r-1)-1$ then $a_{m,n,r} = n$ since one of each type
    special case: $m = n(r-1)+1$ then $a_{m,n,r} = 0$ since one more of any type will violate
- (count partitions of $[n]$ into k blocks)
  the number of partitions of an n-set into k blocks is
  $S(n,k) = \frac{1}{k!}\sum_{i=0}^k (-1)^i\binom{k}{i}(k-i)^n$
  pf: will count $k!S(n,k)$=number of surjective distributions of $[n]$ into k labeled blocks, then divide by $k!$
  note: there are k^n partitions of $[n]$ into k blocks
  let A_j be the set of partitions for which block j is empty
  let $S \subset [n]$ be blocks required to be empty
  then $|\cap_{j\in S}A_j| = (k - |S|)^n$
  PIE: $k!S(n,k) = f(\emptyset) = \sum_{i=0}^k (-1)^i\binom{k}{i}(k-i)^n$
  pf2: generating functions, see §3.3
- (evaluation of sums of form $\sum_{k=0}^n (-1)^k\binom{n}{k}c_k$)
  - when $c_k = 2^{n-k}$
    seek a universe of size 2^n, such as subsets of $[n]$
    seek A_i st $|A_i| = 2^{n-1}$, such as subsets containing i
    then $|\cap_{i\in S}A_i| = 2^{n-|S|}$
    note: the sum counts the subsets of $[n]$ outside all $A_1, ..., A_n$
    so counts the subsets containing no elements
    the only subset is the emptyset
    so $\sum_{k=0}^n (-1)^k\binom{n}{k}2^{n-k} = 1$
    aside: binomial thm says $(-1+2)^n = 1$
  - when $c_k = 1$
    intuition: count by inclusion-exclusion the items in universe U of size 1 that belong to none of n sets each equal to U
    case $n = 0$: sum is 1, since there is one such set
    case $n > 0$: sum is 0
    note: we have counted by PIE and counted directly
  - when $c_k = \binom{m+n-k}{r-k}$
    for universe of size $c_0 = \binom{m+n}{r}$, let U as the family of r-sets in $[m+n]$
    for sets of size $\binom{m+n-1}{r-1}$, let A_i consist of r-sets in $[m+n]$ that use element i
    then for $|S| = k$, $|\cap_{i\in S}A_i| = \binom{m+n-k}{r-k}$ since a member of $\cap_{i\in S}A_i$ has k required elements and the rest from choosing $r-k$ of the remaining $m+n-k$ elements
    so sum is $f(\emptyset)$
    the sum is $\binom{m}{r}$, since the r-sets in none of $A_1, ..., A_n$ are the r-sets in $[m+r]$ that use none of the first n elements
    note: did not need to know the value of the sum beforehand
    note: this generalizes ex3.2.43 which used Snake Oil
- (Eulerian numbers $A(n,k)$)(optional)
  see book for formula for $A(n,k)$
  technique: if know value of sum, divise a set of that size, and use inclusion-exclusion to count the set within a larger universe
- (graph coloring)(optional)
  a proper coloring gives distinct labels (colors, usually in modeled by $[k]$) to adjacent vertices
  see book for the number of proper colorings of a graph using colors in $[k]$
- (a chromatic polynomial)(optinal)
  a chromatic polynomial of a graph is a polynomial in number of colors k which counts the above number of proper colorings
  spanning subgraph see book
  see book

RESTRICTED PERMUTATIONS

Remark.

- recall we restricted multiplicites for selection problems
  here, will restrict positions for arrangement problems

Definition.

- for permutation σ of $[n]$, the permutation matrix is $n \times n$ 0,1-mx with 1 in position i,j iff $\sigma(i) = j$
- board is a $n \times n$ checkered board
- define B as a subset of squares on the board
- independent board positions are at most one every row and column
  model: rooks at those positions arent in each others line of attack
- define $r_k(B)$ as the number of ways to choose k independent positions from B
- the rook polynomial $R_B(x)$ is the generating function $\sum r_k(B)x^k$
- define f_p as the number of elements of U in exactly p of the sets $A_1, ..., A_n \subseteq U$

Proposition.

- (number permutations with general restricted positions)
  Let B set of positions
  restrict permutation mxs with any 1s in position corresponding to B
  the number of allowed permutations is
  $\sum_{k=0}^n (-1)^k r_k(B)(n-k)!$
  pf: let A_i be the set of restricted permutation mxs ie with 1 corresponding to $i \in B$
  note: $|\cap_{i\in S}A_i| = (n - |S|)!$ where set $S \subseteq B$ of independent positions
  so $f(\emptyset) = \sum_{k=0}^n (-1)^k r_k(B)(n-k)!$
- the rook polynomial factors when a board B decomposes into sub-boards B_1 and B_2 occupying distinct rows and cols
  ie $R_B(x) = R_{B_1}(x)R_{B_2}(x)$
  pf: k rooks on B must consist of j on B_1 and $k-j$ on B_2 see book for illustrated example
- (recurrence for $R_B(x)$)
  $R_B(x) = R_{B-s}(x) + xR_{B\cdot s}(x)$ where $B - s$ is subtracting elment s and $B \cdot s$ is subtraction of entire row containing s
  pf: group placements of k rooks by whether they occupy the fixed square s

then $r_k(B) = r_k(B-s) + r_{k-1}(B \cdot s)$
note $r_0(B) = 1$ and for $k > 0$, $r_k(\emptyset) = 0$
multiply by x^k and sum over $k \geq 1$ to yield
$R_B(x) - 1 = R_{B-s}(x) - 1 + xR_{B \cdot s}(x)$

- (general formula, stated very generally)
Let $A_1, ..., A_n \subseteq U$
$\sum_{p=0}^n f_p x^p = \sum_{k=0}^n (x-1)^k h_k$
where recall $h_k := \sum_{S:|S|=k} |\cap_{i \in S} A_i|$
pf: each element $a \in U$ contributes the same to both sides
lhs: contribution of a is x^p where $p = |i \in [n] : a \in A_i|$
rhs: note: $\binom{p}{k}$ is the number of k-subsets of $\{A_1, .., A_n\}$ whose intersection contains a
so a contributes $\binom{p}{k}$ times to h_k
so a contributes $\sum_{k=0}^n (x-1)^k \binom{k}{p}$ to the rhs
which is x^p by the binomial thm
- (rewrite general formula)
$f_p = \sum_{k=p}^n (-1)^{k-p} \binom{k}{p} h_k$
pf: take $\sum_{p=0}^n f_p x^p = \sum_{k=0}^n (x-1)^k h_k$
expand (binom thm) $(x-1)^k = \sum_{p=0}^k \binom{k}{p} (-1)^{k-p} x^p$
and switch order of summation

Remark.

- (general formula in case of permutations with forbidden positions given by board B)
A_i are permutations mxs with 1 in position $i \in B$
f_p is number of permutations having p elements in forbidden positions
$h_k = r_k(B)(n-k)!$ is the number of permutations with $|S| = k$ forbidden positions
so useful formula is
$\sum_{p=0}^n f_p x^p = \sum_{k=0}^n (x-1)^k r_k(B)(n-k)!$
- trick: many problems ask for sum of f_p over all p, so set $x = 1$
- models
  - permutations $\mathfrak{S}_n$ and $n \times n$ permutation mxs
  - permutation mxs and corresponding n indep squares of a board
  - derangements and board with diagonal forbidden
  - independent positions in a board and rooks which arent in each others line of attack

Example.

- (number of derangements of $[n]$)
$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$
pf1: note: forbidden diagonal
$r_k(B) = \binom{n}{k}$ since each subset of the diagonal is indep
sum f_p by setting $x = 0$
pf2: (argument mostly given earlier in §4.1, so I just finished it there)
pf3: prop 2.3.2
pf4: cor 3.3.17
- (expected number of fixed points, "rencontres")
what are the number of fixed points in a random permutation of $[n]$
intuition: randomly extract balls labeled 1 to n from an urn, what is the expected number of times any ith ball is the ith selected
let generating function $N(x) = \sum f_p x^p$ for permutations by number of positions with forbidden elements
use formula with h_k and note $r_k(B) = \binom{n}{k}$
$N(x) = \sum_{k=0}^n (x-1)^k \binom{n}{k} (n-k)! = n! \sum_{k=0}^n \frac{(x-1)^k}{k!}$
get expected weight by differentiate wieght enumerator $N(x)$ (see app3.2.19) (?)
answer: $\frac{N'(1)}{n!}$ which equals 1
method2: use method of ch 14
- count permutations of $[n]$ not having i or $i+1$ in any position i (modulo n)
model: party with n married couples
model with board of forbidden positions of 2 diagonals and one in corner (see image)
use recurrence choosing corner square
see book
method2: model with $0,1$-string of length m with l consecutive 1s
see book for yielding $r_k(B)$

SIGNED INVOLUTIONS

Remark.

- this subsection: a generalization of PIE
- will consider universe X
partitioned into <u>positive part</u> X^+ and <u>negative part</u> X^-
with weight $+1$ for <u>positive elements</u> $x \in X^+$ and -1 for <u>negative elements</u> $y \in X^-$
when sum the weights, unwanted elements cancel since come in $\pm$ pairs, and elements of the desired set have positive weight
- other generalizations we will not cover:
mobius inversion formula of number theory
setting of partially ordered set, see ch15.2

Definition.

- an <u>involution</u> is a permutation whose square is the identity ie cycles have length 1 or 2
- a <u>signed involution</u> on set X (partitioned into X^+,X^-) is an involution τ st every 2-cycle pairs a positive element with a negative element
- denote $F_\tau \subset X^+$ and $G_\tau \subset X^-$ as the fixed points of signed involution τ
- <u>weights</u> of signed involution $w(x) = 1$ for $x \in X^+$ and -1 for $x \in X^-$

Proposition. let signed involution τ on X
$\sum_{x \in X} w(x) = |F_\tau| - |G_\tau|$
pf: contributions from 2-cycles under τ cancel, leaving the fixed points, $|F_\tau|$ with weight $+1$ and $|G_\tau|$ with weight -1

Remark.

- general method:
embed desired set into larger set X
define a "switch" operation on elements of X which applied twice retrieves the original element
the desired fixed points, if any, have no candidate for switching, so the map does not change them
weigh elements which switch with ± 1 so they cancel and fixed points $+1$
sum gives number of fixed points
- signed involutions method allows short proofs that avoid counting

Example.

- let $X = \mathfrak{S}_n$, X^+ are the even permutations, X^- are the odd permutations
  - <u>even</u> (<u>odd</u>) permutation means the number of inversions (swapping order of two elements) is even (odd) (see prop 3.1.15)
  - <u>sign</u> $\text{sign}(\sigma)$ of permutation σ, is $+1$ when σ is even, -1 when σ is odd
  - prop: equivalent def of even/odd: the parity (evenness or oddness) of σ is the parity of the number of transpositions (swap two elements) needed to turn σ into an identity permutation
pf: every transposition changes the number of inversions by an odd amount (?)
  - eg: $X = \mathfrak{S}_3$
even permutations: $X^+ = \{123, 231, 312\}$
odd permutations: $X^- = \{213, 321, 132\}$
note: interchanging first two elements changes parity
  - prop: a signed involution on $\mathfrak{S}_n$ is the involution which transposes the first two elements of each permutations, in word form, in $\mathfrak{S}_n$
  - prop: there are no fixed points (unless $n = 1$)
  - prop: $\mathfrak{S}_n$ has an equal number of even and odd permutations when $n > 1$
  - (application of sign to computing determinant)
let $n \times n$ mx A with $a_{i,j}$ in position i, j
$\det A := \sum_{\sigma \in \mathfrak{S}_n} \text{sign}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$
eg if every entry $a_{i,j} = 1$, then $\det A = 0$
- (PIE by signed involution)
given: $A_1, ..., A_n \subseteq U$
find: $f(\emptyset)$
soln: let $R(x) = \{i \in [n] : x \in A_i\}$
note: each $x \in U$ appears in the inclusion-exclusion formula once for each $S \subseteq R(x)$
group these contributions:
$f(\emptyset) = \sum_{S \subseteq [n]} (-1)^{|S|} |\cap_{i \in S} A_i| = \sum_{x \in U} \sum_{S \subseteq R(x)} (-1)^{|S|} = \sum_{(x,S) \in X} (-1)^{|S|}$
where X is the set of ordered pairs (x, S) st $x \in U$ and $S \subseteq R(x)$
partition X into X^+,X^- by the parity of $|S|$ and weigh elements $+1$ and -1
need signed involution on X st partitions X^+,X^- cancel out except for fixed points $\{(x, \emptyset) : R(x) = \emptyset\} \subset X^+$, then can apply general prop above
choose signed involution which "switches" on whether index of largest A_i containing x is in S:
$$\tau := \begin{cases} (x.S) & R(x) = \emptyset \\ (x, S-i) & i = max(R(x)) \in S \\ (x, S \cup i) & i = max(R(x)) \notin S \end{cases}$$
- for a fixed convex $n - gons$, $e_n - o_n = (-1)^n$ where e_n and o_n count the dissections by nonintersecting diagonals into even or odd number of regions
pf: similar method, see book

DETERMINANTS AND PATHS

Remark. many lattice path problems can be counted by determinant

Definition.

- <u>graph</u> G, <u>vertex set</u> $V(G)$, <u>edge set</u> $E(G)$
- <u>directed graph</u> ("digraph") has <u>edges</u> which are ordered pair (u, v) of vertices u, v
first vertex is <u>tail</u> and second vertex is <u>head</u>
ntn: uv
- a <u>x, y-path</u> P is represented as vertices which can be ordered $v_0, ..., v_k$ with $v_0 = x$ and $v_k = y$, which corresponds to edges $\{v_i v_{i+1} : 0 \leq i \leq k-1\}$
- <u>weighted digraph</u> has weight $w(uv)$ on each edge uv
- <u>weight of subgraph</u> is the product of weights of its edges
- let vertex subsets $X, Y \subseteq V(G)$ where $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_n\}$
the <u>X, Y-path mx</u> has entry i, j as the sum of weights of all paths from x_i to y_j (? what is the weight of a path)
a <u>X, Y-path system</u> $\mathcal{P}$, wrt to permutation $\sigma_\mathcal{P}$ of $[n]$, is set of paths $P_1, ..., P_n$ st P_i is a path from x_i to $y_{\sigma_\mathcal{P}(i)}$
- <u>disjoint path system</u> means paths are pairwise disjoint
- recall $\det A_{n \times n} := \sum_{\sigma \in \mathfrak{S}_n} (\text{sign}\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$ where sign of permutation defined in eg4.1.22

Remark.

- models
  - lattice paths and digraph paths
see book for image of ballot paths as a graph paths
vertices correspond to grid points, edges correspond to allowed transitions
  - every determinant can be interpreted as a signed counting problem on a weighted bipartite digraph
this can be exteded to arbitary directed graphs (?)

- next, formula for determinant can count all disjoint-path problems in all digraphs
  for simplicity, will consider acyclic finite digraphs

Proposition.

- (main thm)
  Let G finite acyclic weighted digraph
  Let X, Y be n-sets of vertices of G
  Let A be X, Y-path mx
  Let $\mathbf{P}$ be the set of disjoint X, Y-path systems
  then $\sum_{\mathcal{P}\in\mathbf{P}}(\text{sign}\sigma_{\mathcal{P}})w(\mathcal{P}) = \det A$
  intuition: determinant sums the signed weights of all disjoint-path systems
  pf: let $\mathbf{Q}$ be the set of all X, Y-path systems, disjoint or not
  seek a signed involution on $\mathbf{Q}$ whose fixed points are the disjoint-path systems and whose systems paired in 2-cycles have the same weight and have associated permutations with opposite sign.
  see book for the signed involution
  so $\sum_{\mathcal{P}\in\mathbf{Q}}(\text{sign}\sigma_{\mathcal{P}})w(\mathcal{P}) = \sum_{\mathcal{P}\in\mathbf{P}}(\text{sign}\sigma_{\mathcal{P}})w(\mathcal{P})$
  next will show that lhs equals $\det A$
  recall: path mx entry $a_{i,j}$ is the sum of the weights of all x_i, y_j-paths
  note: for each σ, each path system $\mathcal{P}$ chooses one $x_i, y_{\sigma(i)}$ path for each i
  $\sum_{\mathcal{P}\text{ wrt }\sigma} w(\mathcal{P}) = \prod_{i=1}^n a_{i,\sigma(i)}$
  since $[\sum_{\text{path systems }\mathcal{P}\text{ of }\sigma} w(\mathcal{P}) = \prod_{i=1}^n \sum_{\text{paths }P\text{ from }i\text{ to }\sigma(i)} w(P)]$ since each path is in some path system
- cor: if every disjoint-path system from $\{p_1, ..., p_n\}$ to $\{q_1, ..., q_n\}$ matches p_i to q_i for all i (so only identity permutation)
  then the number of such disjoint-path systems is the determinant of the mx with (i,j)-entry $h(p_i, q_j)$ where $h((a,b),(c,d)) := \binom{d-b+c-a}{c-a}$
  intuition: number of disjoint paths with startpoints $p_1, ..., p_n$ and respective endpoints $q_1, ..., q_n$ is det of that mx
  pf: main thm applies
  note: the number of paths on digraph, on $\mathbb{Z}^2$ with edges for upward and rightward steps, from point p to point q is $h(p,q)$ since ch1 formula for lattice paths
  note: identity permutation has positive sign
  note: the weight of every path system is 1, since all edge weights are 1
  so the sum of the weights is the number of disjoint-path systems

Example.

- [this is irrelevant, skip] functional digraph (defined in ch1) has each vertex as a tail
  path in a functional digraph f follows repeated iteration of f
- (simplest example of main thm)
  let complete biparpite weighted digraph with X, Y each of size n (see book image of edges $\{x_i y_j : i, j \in [n]\}$, weights $a_{i,j}$)
  count: disjoint path systems
  soln: note: there is one path system $\mathcal{P}_\sigma$ for each permutation σ and each path system is disjoint
  note: $w(\mathcal{P}_\sigma) = \prod_{i=1}^n a_{i,\sigma(i)}$ (?) in this ex
  so $\det A = \sum_{\sigma\in\mathfrak{S}_n}(\text{sign}\sigma)w(\mathcal{P}_\sigma)$
- count disjoint pairs of lattice paths with specified initial points and endpoints
  - inital points $x_1 = (0,1)$, $x_2 = (1,0)$ and endpionts $y_1 = (1,2)$ and $y_2 = (3,2)$
    note: number of paths from x_i to y_j, denoted $a_{i,j}$, fills in mx: $\begin{pmatrix}2 & 4\\ 1 & 6\end{pmatrix}$
    det is 8
    see book for images of the 8 pairs of disjoint paths
  - to count the number of disjoint-path systems that specific number of horizontal steps at some y
    see book for using generating function (?)
  - to count paths avoiding a specified set of lattice points
    see ex58
- properties of determinants following from prop
  - $\det A^T = \det A$
    pf: reverse edges of complete biparpite digraph used above
    the path system associated with σ is associated with σ^{-1} instead
    but they have the same sign, so determinant computation is same (?)
  - mx with linearly dependent rows has determinant 0
    pf: (ex 59)
  - (Cauchy-Binet formula)(optional)
    generalizes determinant product formula to non-square mxs
    see book
- (application)(rhombic tilings of a regular hexigon of side-length n)
  see book

4.2. POLYA-REDFIELD COUNTING.

Remark.

- want to count colorings of an object which are distinguishable from its symmetries
- will count equivalence classes of colorings
- will use simple counting idea: when all equivalence classes have the same size, dividing total number of objects by this size counts the classes
  but might have to adjust the counting st the equivalence classes have same size
- [following defs are for general permutations; we are interested in only symmetries]

Definition.

- a <u>group</u> is a set G together with composition satisfying
  (1) identity: there exists $e \in G$ st $e\pi = \pi = \pi e$ $\forall \pi \in G$
  (2) inverse: for $\pi \in G$ $\exists! \pi^{-1} \in G$ st $\pi\pi^{-1} = e = \pi^{-1}\pi$
  (3) closure: $\pi\sigma \in G$ $\forall \pi, \sigma \in G$
  (4) associative: $(\pi\sigma)\tau = \pi(\sigma\tau)$
- a <u>coloring</u> of set X with set Y of colors is a function $f : X \to Y$
- <u>indistinguishable ("equivalent") colorings</u> f, f' of X wrt G means $\exists \pi \in G$ st $f'(\pi(x)) = f(x)$ $\forall x \in X$
  intuition: can move the object to turn one coloring into the other
- define C as the set of all colorings of X
- ntn: $\pi(u)$ where $\pi \in G$ and $u \in C$, since a permutation on X induces a permutation on each coloring of X
- fix C, G, the <u>orbit</u> of $u \in C$ is $\{\pi(u) : \pi \in G\}$
  intuition: colorings in orbit are indistinguishable
- fix C, G, define $\psi(\pi)$ as the number of elements of C left fixed by $\pi \in G$
- <u>symmetry</u> is a rigid motion that leaves an object occupying the same position, but colors are permuted

Remark.

- [$|C| = |X|^{|Y|}$ ie for fixed object, but we want to count colorings distinguishable wrt symmetry group G]
- the group of symmetries on X and group of symmetries induced on colorings in C have the same structure
  pf: the permutation of each coloring in C induced by the composition of symmetries π, σ is the composition of the permutations of colorings in C induced by π, σ

Proposition.

- indistinguishability is reflexive, symmetric, and transitive ie equivalence relation on the colorings
  intuition: equivalence classes are the colorings indistinguishable by the actions of group of symmetries
  pf: by identity, inverse, and closure properties of G
- orbits partition C, and also, "in the same orbit" is an equivalence relation
  intuition: the orbit of a coloring is the set of colorings equivalent to it
- lem (Lagrange's thm)
  let group G of permutations of C and $u, v \in C$ are in the same orbit
  then $|\{\pi \in G : \pi(u) = v\}| = |\{\pi \in G : \pi(v) = v\}|$
  pf: will establish injections in both directions
  let $A := |\{\pi \in G : \pi(u) = v\}|$ and $B := |\{\pi \in G : \pi(v) = v\}|$
  note: u, v are in same orbit so $\exists \sigma \in G$ st $\sigma(u) = v$
  fix that σ
  note: $\forall \pi \in B$, composition $\pi\sigma \in A$ since takes u to v
  note: $\forall \pi' \in A$, composition $\pi'\sigma^{-1} \in B$ since takes v to v
  both are injective since right multiplying σ^{-1} or σ gives π or π' so $\pi_1\sigma = \pi_2\sigma \Rightarrow \pi_1 = \pi_2$
- (Burnside's lemma)
  the number of equivalence classes of C under the actions of G is $\frac{1}{|G|}\sum_{\pi\in G}\psi(\pi)$
  pf: the following equalities will be explained
  $\sum_E 1 = \frac{1}{|G|}\sum_E |G| = \frac{1}{|G|}\sum_E\sum_{v\in E}|\{\pi : \pi(u_E) = \frac{1}{|G|}\sum_{v\in C}\phi(v) = \frac{1}{|G|}\sum_{\pi\in G}\psi(\pi)$
  want to count 1 for each class E
  this equals counting $|G|$ for each E and dividing by $|G|$
  note: $|G| = \sum_{v\in E}|\{\pi : \pi(u_E) = v\}|$ since we group each $\pi \in G$ by $\pi(u_E) \in E$ where $u_E \in E$ is a chosen distinguised element from each class $E \subseteq C$
  use Lagrange's thm to group by $v \in C$, where $\phi(v) :=$number of operators fixing v
  $\sum_{v\in C}\phi(v)$ counts of pairs (v, π) st $\pi(v) = v$ ie fixed points
  $\sum_{\pi\in G}\psi(\pi)$ counts the same set, but grouped by operators
- lem: Let X colored by k colors
  $\psi(\pi)$ (number of colorings left fixed by π) is k^t where t is the number of cycles (recall cycle representation of π on X is parenthesized listing of cycles) created by π on X
  pf: note: each cycle has same color
  there are k choices for each cycle and t cycles

Remark.

- the final step in Burnside's lemma enables convenient computation
- technique: raise the count so that each class is counted equally often, then divide by that number
- if elements in G map coloring f to distinct colorings
  then orbit has size $|G|$, and dividing by $|G|$ counts it once

Example.

- a regular n-gon can be rotated by any multiple of $2\pi/n$ radians and flipped about any of n axes
  so $2n$ symmetries in space
  note: these operations permute both the vertices and edges
- introductory examples
  - given baton partitioned into n bands of equal length
    note: can flip baton without changing it
    count: number of ways to color the bands with k colors
    k^n is number of colorings of fixed object

$k^{\lceil n/2 \rceil}$ is number of symmetric colorings, since paint half, and flip
k^n counts symmetric colorings once and non-symmetric colorings twice
so add number of symmetric colorings and divide by 2
so $\frac{k^n + k^{\lceil n/2 \rceil}}{2}$ colorings

– count: number of ways to paint corners of triangle with k colors
intuition: clown pinning k types of flowers to tri-cornered hat
there are k^3 colorings of the fixed object
k^3 counts the k monochromatic colorings once each and all others three times each so add in two times the monochromatic colorings, and divide by 3
so $\frac{k^2+2k}{3}$ colorings

- necklaces
  – consider necklace of n beads drawn from k types (colors)
  symmetries: cyclic rotations and flips
  fixed necklace has k^n ways
  note: the k monochromatic classes have size one, and classes with periodic lists are also smaller than the others, so must adjust counting st classes have equal size
  – 4-bead necklace
  count: number of colorings of 4-bead neckace with k colors
  soln: model with symmetries of corners of a square
  note: 8 symmetries: 4 rotations and 4 reflections
  rotations: 0,90,180,270 degrees have 4,1,2,1 cycles respectively
  reflections: horizontal and vertical flips have two cycles of length 2, diagonal flips have one cycle of length 2, two of length 1
  in summary $1, 2, 3, 2$ symmetries have $4, 3, 2, 1$ cycles, respectively
  so $\frac{1}{8}(k^4 + 2k^3 + 3k^2 + 2k)$ distinguishable colorings
- count isomorphism classes of simple graphs with n vertices
  note: there are $2^{\binom{n}{2}}$ graphs with vertex set $[n]$
  note: two graphs are isomorphic when a vertex permutation turns one into the other
  note: isomorphism is an equivalence relation since can be compounded and inverted
  colors: vertex pairs are colored "edge" or "non-edge"

THE PATTERN INVENTORY

Remark. want to count the distinguishable patterns in which each color appears a specified number of times

Definition.
- a cycle representation of permutation π is a parenthesized listings of cycles by their elements in order
- a cycle structure of π is product $\prod x_j^{e_j}$ where π has e_j cycles of length j
  note: x_j represents a cycle of length j
  intuition: represent lengths of cycles of π as a monomial
- the cycle index $Z_G(x_1, x_2, ...)$ of group G of permutations is $\frac{1}{|G|}$ times the generating function summing $\prod x_j^{e_j}$ with coefficients as number of elements of G with that cycle structure
  note: the cycle index is for permutations of the object being colored
- let $Y = \{y_1, ..., y_k\}$ be a set of colors
- the pattern inventory of colorings of X is the OGF in which the coef of $\prod y_i^{e_i}$ is the number of equivalence classes of colorings with color y_i on e_i elements of X
  pattern inventory is cycle index with $x_j = y_1^j + y_2^j + y_3^j + ...$
  intuition: records how many times each color is used ie records the options instead of just the number of options
- a weight enumerator for colors is a generating function $w(y)$ whose coef of y^i is the number of colors with weight i (?)

Remark.
- let cycle structure $\prod x_j^{e_j}$ of a permutation of X then $\sum j e_j = |X|$ since each element of X appears in one cycle
- general method:
  let set X colored by Y and moved by symmetry group G
  write cycle structure for each $\pi \in G$
  sum to get cycle index
  plug $x_j = k$ into cycle index to to count the equivalence classes of colorings when k colors are available
  plug $x_j = \sum_{i=1}^k y_i^j$ into cycle index to get the pattern inventory listing equivalence classes by how many times each color is used

Proposition.
- let group G of permutations on X
  the number of distinguishable k-colorings of X under G is $Z_G(k, k, k, ...)$
  pf: set each $x_j = k$
  this accumulates $\psi(\pi)$ for each $\pi \in G$
  divide by $|G|$ to count one for each class
- (Polya's thm)
  (?) Let group G of permutations on X
  let weight enumerator w for color options
  The OGF for equivalence classes of colorings of X under the action of G, enumerated by weight, is obtained from the cycle index Z_G be setting each variable x_j to the expression obtained from w by replacing each variable with its jth power
  pf: see book

Example.
- cycle indices
  – color vertices of a rotating triangle
  recall: 1 symmetry has 3 cycles of length 1, and 2 symmetries has 1 cycle of length 3
  cycle index: $\frac{1}{3}(x_1^3 + 2x_3^1)$
  – color vertices of square tumbling in space
  recall: 1 summetry has 4 cycles of length 1, 2 symmetries have 1 cycle of length 4, 3 symmetries have 2 cycles of length 2, and 2 symmetries have 2 cycles of length 1 and 1 cycle of length 2
  cycle index: $\frac{1}{8}(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2 x_2)$
- pattern inventory
  – consider 4-bead necklaces colored with blue and red
  plug $x_j = b^j + r^j$ pattern inventory: $\frac{1}{8}((b+r)^4 + 2(b^4 + r^4) + 3(b^2 + r^2)^2 + 2(b+r)^2(b^2 + r^2)) = b^4 + b^3 r + 2b^2 r^2 + br^3 + r^4$
  note $2b^2r^2$ represents 2 beads of each color in alternating and non-alternating patterns
  – consider a cube, 24 symmetries
  symmetries: identity, three rotations for each pair of opposite faces, one for each pair of opposite edges, and two for each pair of opposite vertices there are isomorphic groups with symmetries of vertices, faces, and edges
  but the corresponding cycle structures are different since the set X being colored is different
  see book for table of cycle structures and resulting cycle indices for colorings of vertices, faces, and edges
- weight enumerator examples
  – (application)(number of isomorphic classes of n-vertex graphs)
  let X=the set of vertex pairs
  colors: each vertices pair is colored "edge" and "non-edge"
  the group of permutations of vertices, $\mathfrak{S}_n$, induces $n!$ permutations on X
  these induced permutations is the pair group $\mathfrak{S}_n^{(2)}$ which turn one coloring into another
  for a vertex set, isomorphic graphs G,H means some permutation turns one into the other ie $uv \in E(G)$ iff $\pi(u)\pi(v) \in E(H)$
  claim: the number of isomorphism classes of n-vertex graphs with m edges is the coef of y^m in the OGF by plugging $x_j = 1_y^j$ into cycle index $Z_{\mathfrak{S}_n}^{(2)}$
  pf: will do case $n = 4$, other n are similar
  see book for table for computation of $Z_{\mathfrak{S}_4^{(2)}}$
  see book for cycle index of $\mathfrak{S}_4^{(2)}$
  plugging $x_i = 2$ into cycle index yields 11 isomorphism classes
  see book for images of isomorphism classes
  setting $x_j = 1 + y^j$ models ...
  – paint faces of cube
  see book

CLASSICAL CYCLE INDICES

Example.
- the cyclic group
- crowns with r missing jewels