PREFACE

Remark.

- Real analysis started as theory of functions on $\mathbb{R}$, but exanded to include many related theories
- Prereqs
  - classical theory of functions of a real variable (limits and continuity, differantiation and Riemann integration, infinite series, uniform convergence, metric spaces)
    see Baby Rudin
  - arithmetic of complex numbers, including properties of $e^{ix} = e^x(\cos x + i \sin x)$
    see Baby Rudin
  - elementary set theory
    see chapter 0
  - linear algebra (vector spaces, linear mappings, determinants)
- chapters
  - standard real analysis course:
    * 1,2,3: measures, integration, differentiation
    * 4,5: Topology and functional analysis
      mostly indep of ch1,2,3
    * 6,7: L^p-spaces, Radon measures
  - extra topics
    * 8,9: Fourier analysis, distribution theory
    * 10,11: Probability theory, more measures and integrals
      indep of 8,9, except §8.6 used in ch10

0. PROLOGUE

Remark. this chapter is a terse (dense) reference

0.1. THE LANGUAGE OF SET THEORY.

0.1.1. *NUMBER SYSTEMS.*

Definition. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

0.1.2. *LOGIC.*

Definition.

- contrapositive: $(-B \Rightarrow -A) \Leftrightarrow (A \Rightarrow B)$
- reductio ad absurdum ("proof by contradiction"): $(A, -B \Rightarrow \text{condtradiction}) \Rightarrow (A \Rightarrow B)$

0.1.3. *SETS.*

Definition.

- set X is [?]
- inclusion $A \subset X$ ("$\subseteq$"): $[x \in E \Rightarrow x \in X]$
- empty set $\emptyset$
- power set $\mathcal{P}(X) := \{E : E \subset X\}$
- family ("collection") of sets is a set of sets
- Let $\mathcal{E}$ a family of sets
  union $\cup_{E \in \mathcal{E}} E := \{x : x \in E \text{ for some } E \in \mathcal{E}\}$
  intersection $\cap_{E \in \mathcal{E}} E := \{x : x \in E \text{ for all } E \in \mathcal{E}\}$
- indexed family of sets $\mathcal{E} := \{E_\alpha : \alpha \in A\}$
  ntn: $\{E_\alpha\}_{\alpha \in A}$
- disjoint collection of sets is a collection of disjoint sets
- disjoint union of sets is a union of disjoint sets
- let $\{E_n\}_{n \in \mathbb{N}}$
  limit superior $\limsup E_n := \cap_{k=1}^\infty \cup_{n=k}^\infty E_n = \{x : x \in E_n \text{ for infinitely many } n\}$
  limit inferior $\liminf E_n := \cup_{k=1}^\infty \cap_{n=k}^\infty E_n = \{x : x \in E_n \text{ all but finitely many } n\}$
- set difference $E \setminus F := \{x : x \in E \text{ and } x \notin F\}$
- set symmetric difference $E \triangle F := (E \setminus F) \cup (F \setminus E)$
- complement of E wrt X $E^c = X \setminus E$

Claim. *(DeMorgan's laws)*
$(\cup_{\alpha \in A} E_\alpha)^c = \cap_{\alpha \in A} E_\alpha^c$
$(\cap_{\alpha \in A} E_\alpha)^c = \cup_{\alpha \in A} E_\alpha^c$

[RELATIONS]

Definition.

- cartesian product $X \times Y := \{(x, y) : x \in X \text{ and } y \in Y\}$
  ordered pair (x, y) is ?
- relation from X to Y is $R \subset X \times Y$
  ntn: xRy means $(x, y) \in R$
- relation on X is $R \subset X \times X$
- equivalence relation R on X:
  $xRx \ \forall x \in X$
  $xRy \Leftrightarrow yRx$
  xRz whenever xRy and yRz for some y
  equivalence class of y is $\{x : xRy\}$
  note: X is a disjoint union of equiv classes

[MAPPING RELATION]

Definition.

- mapping: relation from X to Y st $\forall x \in X$ $\exists! y \in Y$ st xRy
  ntn: $f : X \to Y$, $y = f(x)$
  function is a mapping to $\mathbb{C}$
- composition of mappings $f : X \to Y$, $g : Y \to Z$ is mapping $g \circ f : X \to Z$
  ntn: $g \circ f(x) = g(f(x))$
- image of $D \subset X$ under $f : X \to Y$ is $f(D) := \{f(x) : x \in D\}$
- inverse image of $E \subset Y$ under $f : X \to Y$ is $f^{-1}(E) := \{x : f(x) \in E\}$

Claim.

- *$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ commutes with union, intersection, and complement:*
  $f^{-1}(\cup_{\alpha \in A} E_\alpha) = \cup_{\alpha \in A} f^{-1}(E_\alpha)$
  $f^{-1}(\cap_{\alpha \in A} E_\alpha) = \cap_{\alpha \in A} f^{-1}(E_\alpha)$
  $f^{-1}(E_\alpha^c) = \left(f^{-1}(E_\alpha)\right)^c$
- *$f : \mathcal{P}(X) \to \mathcal{P}(Y)$ commutes with union, but not always with intersection, and complement*

Definition. let mapping $f : X \to Y$

- domain X range $f(X)$
- injective f means $f(x_1) = f(x_2)$ only when $x_1 = x_2$
  surjective f means $f(X) = Y$
  bijective f means injective and surjective

Claim. *If $f : X \to Y$ bijection*
Then $\exists$ inverse $f^{-1} : Y \to X$ st $f \circ f^{-1}$ and $f^{-1} \circ f$ are the identity mappings on X and Y

Definition. restriction of $f : X \to Y$ to A is $(f|A) : A \to Y$, $(f|A)(x) = f(x)$ for $x \in A$

Definition.

- sequence is a mapping $f : \mathbb{N} \to X$
  ntn: $\{x_n\}_1^\infty$ where $x_n := f(n)$, so can treat as subset of X
- finite sequence is $f : \{1, 2, ..., n\} \to X$
- subsequence of $f : \mathbb{N} \to X$ is $f \circ g$ where $g : \mathbb{N} \to \mathbb{N}$ and $g(n) < g(m)$ when $n < m$

Definition.

- Cartesian product of infinite families of sets:
  (since earlier def is awkward for infinite families of sets)
  $\prod_{\alpha \in A} X_\alpha$ of indexed family of sets $\{X_\alpha\}_{\alpha \in A}$ is the set of all maps $f : A \to \cup_{\alpha \in A} X_\alpha$ st $f(\alpha) \in X_\alpha$ for all $\alpha \in A$
- α^{th} projection ("coordinate map") $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$, $\pi_\alpha(f) := f(\alpha)$
  the α^{th} coordinate of f is $f(\alpha)$
  ntn: write x and x_α instead of f and $f(\alpha)$
- if $Y = X_\alpha$ for all α:
  $Y^A := \prod_{\alpha \in A} X_\alpha$ is the set of all mappings from A to Y
  ntn: if $A = \{1, 2, ..., n\}$ then Y^A written Y^n=the set of ordered n-tuples of elements of Y

0.2. ORDER THEORY.

Remark. [want a notion of "this precedes that"]

Definition.

- partial order on $X \neq \emptyset$ is a relation R on X st
  reflexive: $xRx \ \forall x$
  anti-symmetric: xRy and $yRx \Rightarrow x = y$
  transitive: xRy and $yRz \Rightarrow xRz$
  ntn: $\leq$
  image: [finite poset as a Hasse diagram]
  [poset is a partially ordered set]
- linear ("total") order also satisfies
  totality: $x, y \in X \Rightarrow xRy$ or yRx
  [chain is a linearly ordered set]

Example.

- partial order: sets by $\subset$
- total order: $\mathbb{R}$ by $\leq$
- [identity order = is the only patial order and equivalence relation]

Claim.

- *a partial ordering on X naturally induces a partial ordering on every nonempty subset of X*
- *[a total ordering naturally induces a total order on nonempty subsets]*

Definition.

- [let function $f : X \to Y$ where $(X, \leq_X)$ and $(Y, \leq_Y)$ are partial orders
  order preserving ("monotone") means $(x_1 \leq_X x_2) \Rightarrow (f(x_1) \leq_Y f(x_2))$
  order reversing ("antitone") means $(x_1 \leq_X x_2) \Rightarrow (f(x_2) \leq_Y f(x_1))$
  order reflecting means $(f(x_1) \leq_Y f(x_2)) \Rightarrow (x_1 \leq_X x_2)$
  order embedding means both order preserving and order reflecting]
- order isomorphic partially ordered sets X, Y: $\exists$ bijection $f : X \to Y$ st $x_1 \leq_X x_2$ iff $f(x_1) \leq_Y f(x_2)$

Remark. [some elements play special roles]

Definition.

- [the least (greatest) element of partial order $(X, \leq)$ is $x \in X$ st $x \leq y$ $(y \leq x)$ $\forall y \in X$]
- a minimal (maximial) element of partial order $(X, \leq)$ is $x \in X$ st the only $y \in X$ satisfying $x \leq y$ $(y \leq x)$ is x itself

Example.

- [1 is minimal (and least) element for positive integers]
- [$\emptyset$ is minimal (and least) element for the subset order]
- [an element can be both maximal and minimal]

Claim.

- *[least and greatest elements need not exist, but are unique if they do]*
- *maximal and minimal elements may not exist and need not be unique unless the ordering is linear*

Proof. eg [don't exist for open interval of $\mathbb{R}$]
eg [maximal element doesnt exist for the set of all finite subsets of a given infinite set]

Definition.

- an upper (lower) bound for $E \subset X$ is $x \in X$ st $y \leq x$ $(x \leq y)$ for all $y \in E$
- [least upper bound] ("lub", "supremum", "join")
  [greatest lower bound ("glb", "infimum", "meet")]
- [complete partial order means lub and glb exist for subsets eg $(\mathbb{R}, \leq)$]

Example. [consider $(\mathbb{N}, |)$ where $|$ is the divisibility relation eg $2|4$
lub: least common multiple
glb: greatest common divisor]

Claim.

- *an upper bound of E need not be in E*
- *unless E is linearly ordered, a maximal element of E need not be an upper bound for E*

Proof. left to reader

Definition. <u>well ordered</u> X by $\leq$ means linearly ordered and every subset of X has a (necessarily unique) minimal element
ntn: $\leq$

Example. $\mathbb{N}$ is well ordered by the usual $<$ (?)

Remark. next: a fundamental concept of set theory, and some consequences

Theorem. *(Hausdorff Maximum Principle)*
Every partially ordered set has a maximal linearly ordered subset
ie if X is partially ordered by $\leq$, then $\exists E \subset X$ lineary ordered by $\leq$ st no subset of X that properly includes E is linearly ordered by $\leq$

Theorem. *(Zorn's lemma)*
if X is partially ordered set and every linearly ordered subset of X has an upper bound
then X has a maximal element
ie maximal element exists under certain conditions

Proof. $\Leftrightarrow$ Hausdorff Maximum Principle
($\Leftarrow$) an upper bound for a maximal linearly ordered subset of X is a maximal element of X
($\Rightarrow$) apply Zorn's lemma to the collection of linearly ordered subsets of X, which is partially ordered by inclusion

Theorem. *(Well Ordering Principle)*
every nonempty set X can be well ordered

Proof. short, see book

Theorem. *(Axiom of Choice)*
If $\{X_\alpha\}_{\alpha\in A}$ is a nonempty collection of nonempty sets
then $\prod_{\alpha\in A} X_\alpha$ is nonempty

Proof. let $X = \cup_{\alpha\in A} X_\alpha$
pick a well-ordering on X
for $\alpha \in A$, let $f(\alpha)$ be the maximal element of X_α
then $f \in \prod_{\alpha\in A} X_\alpha$

Corollary. *if $\{X_\alpha\}_{\alpha\in A}$ is a disjoint collection of nonempty sets*
then $\exists Y \subset \cup_{\alpha\in A} X_\alpha$ st $Y \cap X_\alpha$ contains exactly one element for each $\alpha \in A$

Proof. take $Y = f(A)$ where $f \in \prod_{\alpha\in A} X_\alpha$

Remark. we have shown Hausdorff Maximum Principle $\Rightarrow$ Axiom of Choice
can show $\Leftarrow$ as well

0.3. CARDINALITY.

Definition. Let X, Y nonempty sets

- <u>card(X) $\leq$card(Y)</u>: $\exists f : X \to Y$ injective
- <u>card(X) $\geq$card(Y)</u>: $\exists f : X \to Y$ surjective
- <u>card(X) =card(Y)</u>: $\exists f : X \to Y$ bijective
- <u>card(X) $<$card(Y)</u>: $\exists f : X \to Y$ injective but not bijective
- <u>card(X) $>$card(Y)</u>: $\exists f : X \to Y$ surjective but not bijective
- extend to $\emptyset$ with card(X) $>$card($\emptyset$) and card($\emptyset$) $<$card(X) for all $X \neq \emptyset$
- not "card(X(" has no meaning. But can define to be size for finite set.

Remark.

- not "card(X(" has no meaning. But can define to be size for finite set:
  card(X) $= n$ means card(X) =card$\{1, ..., n\}$
- extend to $\emptyset$ with card(X) $>$card($\emptyset$) and card($\emptyset$) $<$card(X) for all $X \neq \emptyset$
- will assume all sets are nonempty to avoid arguments for case $\emptyset$

Proposition. *card(X) $\leq$card(Y) iff card(Y) $\geq$card(X)*

Proof. book

Proposition. *$\forall X, Y$, either card(X) $\leq$card(Y) or card(X) $\geq$card(Y)*

Proof. book

Theorem. *(Schroder-Bernstein)*
If card(X) $\leq$card(Y) and card(X) $\geq$card(Y)
then card(Y) =card(X)

Proof. book

Proposition. *$\forall X$, card(X) $<$card($\mathcal{P}(X)$)*

Proof. book

Definition.

- <u>countable</u> ("denumerable") set X: card(X) $\leq$card($\mathbb{N}$)
- <u>countably infinite</u> set: countable but not finite

Proposition.

a. X, Y countable $\Rightarrow$ so is $X \times Y$
b. A countable and X_α countable for every $\alpha \in A$ then $\cup_{\alpha\in A} X_\alpha$ is ctble
c. X countable infinite $\Rightarrow$ card(X) =card($\mathbb{N}$)

Proof. book

Corollary. *$\mathbb{Z}$ and $\mathbb{Q}$ are ctble*

Definition. <u>cardinality of the continuum</u> of X means card(X) =card($\mathbb{R}$)
ntn: $\mathfrak{c}$ instead of card($\mathbb{R}$)

Proposition. *card($\mathcal{P}(\mathbb{N})$) $= \mathfrak{c}$*

Corollary. *card(X) $\geq \mathfrak{c} \Rightarrow X$ is uncountable*

Remark. the converse is the continuum hypothesis
note: validity of continuum hypothesis is a famous undecidable problem in set theory

Proposition.

a. card(X) $\leq \mathfrak{c}$ and card(Y) $\leq \mathfrak{c} \Rightarrow$ card($X \times Y$) $\leq \mathfrak{c}$
b. card(X) $\leq \mathfrak{c}$ and card(X_α) $\leq \mathfrak{c}$ $\forall\alpha \in A \Rightarrow$ card($\cup_{\alpha\in A} X_\alpha$) $\leq \mathfrak{c}$

Proof. book

0.4. MORE ABOUT WELL ORDERED SETS.

Definition. Let X well-ordered

- <u>infimum</u> of nonempty $A \subset X$ is its maximal lower bound
  <u>supremum</u> of A, is a mimumal upper bound ($\exists$ it is bdd above)
  ntn: $\inf A$, $\sup A$
- the <u>initial segment</u> of $x \in X$ is $I_x := \{y \in X : y < x\}$ the <u>predecessors</u> of $x \in X$ are the elements in I_x

Remark. the principle of mathematical induction relies on $\mathbb{N}$ being well-ordered
next will extend induction to arbitrary well-ordered sets

Proposition. *(The Principle of Transifinite Induction)*
X well-ordered, $A \subset X$, $x \in A$ whenever $I_x \subset A$
then $A = X$

Proof. towards contradiction, assume $X \neq A$
let $x = \inf(A \setminus A)$
then $I_x \subset A$, but $x \notin A$

Proposition. *X well ordered, $A \subset X$*
then $\cup_{x\in A} I_x$ is either an initial segment or X itself

Proof. book

Proposition. *X, Y well ordered*
then either X order isomorphic to Y or X order isomorphic to an initial segment of Y, or Y order isomorphic to an initial segment of X

Proof. book

Proposition. *$\exists$ an unctble well ordered set Ω st I_x is ctble for each $x \in \Omega$*
if Ω' is another set with the same properties, then Ω and Ω' are order isomorphic

Proof. book

Definition. the <u>set of ctble ordinals</u> is Ω above

Proposition. *every ctble subset of Ω has an upper bound*

Proof. book

Example. $\mathbb{N}$ is order isomorphic with subset $I_\omega \subset \Omega$, where ω is the minimal element of Ω st I_ω is finite.
pf: set $f(1) = \inf \Omega$
inductively set $f(n) = \inf(\Omega \setminus \{f(1), ..., f(n-1)\})$

Definition. <u>first uncountable ordinal</u> ω_1
$\Omega^* := \Omega \cup \{\omega_1\}$ is extension of Ω with an extra element ω_1.
and extend the ordering on Ω to Ω^* st $x < \omega_1$ $\forall x \in \Omega$
ntn: ω_1 is often written Ω since ω_1 is generall taken to be the set of ctble ordinals itself.

0.5. THE EXTENDED REAL NUMBER SYSTEM.

Definition.

- <u>complete</u> set X:
  for every $A \subset X$, $\inf A, \sup A \in X$
  ($\Leftrightarrow$ every sequence $\{x_n\} \subset \overline{X}$ has $\liminf x_n$ and $\limsup x_n$ in X)
- let sequence $\{x_n\}$
  $\liminf x_n := \inf_{k\geq 1} \left(\sup_{n\geq k} x_n\right)$
  $\limsup x_n := \sup_{k\geq 1} \left(\inf_{n\geq k} x_n\right)$
  intuition: inf and sup are generalizations of max and min
- <u>convergent sequence</u> $\{x_n\}$ means $\liminf x_n = \limsup x_n$
  <u>limit</u> is the comon value

Definition.

- <u>extended real number system</u> $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$
  where extend ordering $<$ st $-\infty < x < \infty$ $\forall x \in \mathbb{R}$
  extend arithmetic: $\infty + \infty := \infty$, $-\infty - \infty := -\infty$, $0 \cdot (\pm\infty) := 0$
  $x \pm \infty := \pm\infty (x \in \mathbb{R})$, $x \cdot (\pm\infty) := \pm\infty (x > 0)$, $x \cdot (\pm\infty) := \mp\infty (x < 0)$
  note: $\infty - \infty$ undefined
- <u>intervals</u> in *overline*$\mathbb{R}$ for $-\infty \leq a < b \leq \infty$:
  $(a, b) = \{x : a < x < b\}$
  $[a, b) = \{x : a \leq x < b\}$
  $(a, b] = \{x : a < x \leq b\}$
  $[a, b] = \{x : a \leq x \leq b\}$

Claim. *$\overline{\mathbb{R}}$ is complete*

Definition. <u>unctble sum of nonnegative numbers</u>
let $f : X \to [0, \infty]$ where X arbitary
define $\sum_{x\in X} f(x) := \sup \{\sum_{x\in F} f(x) : F \subset X, F$ is fin
intuition: later will define this as integral of f wrt counting measure

Proposition. *let $f : X \to [0, \infty]$, $A = \{x : f(x) > 0\}$*
if A unctble: $\sum_{x\in X} f(x) = \infty$
if A ctbly infinite: $\sum_{x\in X} f(x) = \sum_1^\infty f(g(n))$ where $g : \mathbb{N} \to A$ is any bijection

Proof. book

Definition.

- consider $f : X \to \overline{R}$ where X arbitrary
  ntn: $f < g$ means $f(x) < g(x)$ $\forall x$
  ntn: $\max(f, g)$ at x means $\max(f(x), g(x))$
- consider $f : X \subset \overline{\mathbb{R}} \to \overline{R}$
  increasing means $f(x) \le f(y)$ whenever $x \le y$
  strictly increasing means $f(x) < f(y)$ whenever $x < y$
  decreasing similar
  monotone means either increasing or decreasing
- conider $f : \mathbb{R} \to \mathbb{R}$
  right-hand limit $f(a+) = \lim_{x \searrow a} f(x) := \inf_{x>a} f(x)$
  left-hand limit $f(a-) = \lim_{x \nearrow a} f(x) := \sup_{x<a} f(x)$
  note: $f(\infty) = \sup_{a\in\mathbb{R}} f(a)$, $f(-\infty) = \inf_{a\in\mathbb{R}} f(a)$
- conider $f : \mathbb{R} \to \mathbb{R}$
  right continuous f means $f(a) = f(a+)$ $\forall a \in \mathbb{R}$
  left continuous f means $f(a) = f(a-)$ $\forall a \in \mathbb{R}$
- modulus of $x \in \mathbb{C}$ is $|x| := |a + ib| = \sqrt{a^2 + b^2}$
  euclidean norm of $x \in \mathbb{C}^n$ is $|x| := \sqrt{\left(\sum_1^n a_j^2\right)}$
- open set $U \subset \mathbb{R}$ means $\forall x \in U$, U includes an interval centered at x

Proposition. *every open set in $\mathbb{R}$ is a ctble disjoint union of open intervals*

0.6. METRIC SPACES.

Definition.

- a metric on X is $\rho : X \times X \to [0, \infty)$ st
  (nonnegative) $\rho(x, y) = 0$ iff $x = y$
  (symmetric) $\rho(x, y) = \rho(y, x)$ $\forall x, y \in X$
  (triangle $\le$) $\rho(x, z) \le \rho(x, y) + \rho(y, z)$ $\forall x, y, z \in X$
  intuition: $\rho(x, y)$ =distance between x and y
- metric space: (X, ρ)

Example. • the Euclidean distance $\rho(x, y) = |x - y|$ is a metric on $\mathbb{R} * n$

- $\rho_1(f, g) = \int_0^1 |f(x) - g(x)|\,\mathrm{d}x$ and $\rho_\infty(f, g) = \sup_{0\le x\le 1} |f(x) - g(x)|$ are metrics on the space of continuous functions on $[0, 1]$
- let metric ρ on X
  then $\rho|(A \times A)$ is a metric on $A \subset X$
- product metric of metric spaces (X_1, ρ_1) and (X_2, ρ_2) is metric ρ on $X_1 \times X_2$, $\rho((x_1, x_2), (y_1, y_2)) = \max(\rho_1(x_1, y_1), \rho_2(x_2, y_2))$
- other metrics on $X_1 \times X_2$ are:
  $\rho_1(x_1, y_1) + \rho_2(x_2, y_2)$
  $\left(\rho_1(x_1, y_1)^2 + \rho_2(x_2, y_2)^2\right)^{1/2}$

Definition. Let metric space (X, ρ)

- the open ball of radius r about x is $B(r, x) := \{y \in X : \rho(x, y) < r\}$
- open $E \subset X$ means for every $x \in E$, $\exists r > 0$ st $B(r, x) \subset E$
- closed $E \subset X$ means its compliment is open
- the interior of $E \subset X$, E^o is the largest open set contained in E, ie the union of all open sets in E
  the closure of $E \subset X$, $\overline{E}$, is the smallest closed set containing E, ie the intersection of all closed sets containing E
- dense set $E \subset X$ means $\overline{E} = X$
  nowhere dense set $E \subset X$ means $\overline{E}$ has an empty interior
- seperable set X means iti has a ctble dense subset
- sequence $\{x_n\} \subset X$ converges to $x \in X$ means $\lim_{n\to\infty} \rho(x, x_n) = 0$
  ntn: $x_n \to x$ or $\lim_{n\to\infty} x_n = x$

Claim.

- *every ball is open*
- *$\emptyset$ and set X (wrt (X, ρ)) are both open and closed*
- *the union of any family of open sets is open*
  the intersection of any family of closed sets is closed
- *the intersection of any finite family of open sets is open*
- *the union of any finite family of closed sets is closed*
- *$\mathbb{R}^n$ is seperable since $\mathbb{Q}^n$ is a ctble dense subset*

Proposition. *if X metric space, $E \subset X$, and $x \in X$*
then the following are equivalent

- *$x \in \overline{E}$*
- *$B(r, x) \cap E \ne 0$ $\forall r > 0$*
- *$\exists$ sequence $\{x_n\} \subset E$ that converges to x*

Proof. book

Definition. let metric spaces $(X_1, \rho_1), (X_2, \rho_2)$

- continuous map $f : X_1 \to X_2$ at $x \in X_1$ means $\forall \epsilon > 0$ $\exists \delta > 0$ st $\rho_2(f(y), f(x)) < \epsilon$ whenever $\rho_1(x, y) < \delta$
  ie $f^{-1}(B(\epsilon, f(x))) \supset B(\delta, x)$
- continuous map means continuous at all points x
- uniformly continuous map means continous and δ in the map can be chosen independently of the point x

Proposition. *$f : X_1 \to X_2$ is continuous iff $f^{-1}(U)$ is open in X_1 for every open $U \subset X_2$*

Proof. book

Definition. • Cauchy sequence $\{x_n\}$ in metric space (X, ρ) means $\rho(x_n, x_m) \to 0$ as $m, n \to 0$

- complete subset $E \subset X$ means every Cauchy sequence in E converges and its limit is in E

Example. $\mathbb{R}^n$ with Euclidean metric is complete
$\mathbb{Q}^n$ is not

Proposition. *A closed subset of a complete metric space is complete*
A complete subset of an arbitrary metric space is closed

Proof. book

Definition. Let metric space (X, ρ)

- $\rho(x, E) := \inf\{rho(x, y) : y \in E\}$ is the distance from a point to a set $E \subset X$
- $\rho(F, E) := \inf\{\rho(x, y) : x \in E, y \in F\} = \inf\{rho(x, F) : x \in E\}$ is the distance between setts $E, F \subset X$
- the diameter of $E \subset E$ is $\mathrm{diam}(E) = \sup\{\rho(x, y) : x, y \in E\}$
- bounded $E \subset X$ means $\mathrm{diam}(E) < \infty$
- a cover of $E \subset X$ is family of sets $\{V_\alpha\}_{\alpha\in A}$ st $E \subset \cup_{\alpha\in A} V_\alpha$
- totally bounded set $E \subset X$ means $\forall \epsilon > 0$ E can be covered by finitely many balls of radius ϵ

Claim.

- *$\rho(x, E) = 0$ iff $x = \overline{E}$*
- *every totally bdd set is bdd*
  the converse is false in general
- *if E is bdd, so is $\overline{E}$*

Proof. book

Theorem. *Let metric space (X, ρ), $E \subset X$*
the following are equivalent:

a. E is complete and totally bdd
b. (Bolzano-Weierstrass property) every sequence in E has a subsequence that converges to a point of E
c. (Heine-Borel property) If $\{V_\alpha\}_{\alpha\in A}$ covers E by open sets
then there is a finite set $F \subset A$ st $\{V_\alpha\}_{\alpha\in A}$ covers E

Proof. book, 1 page

Definition. compact set E means satisfies a,b,c in above thm

Proposition.

- *every compact set is closed and bdd*
- *converse is true in $\mathbb{R}^n$*

Proof. book

Definition. equivalent metrics ρ_1, ρ_2 on set X means $C\rho_1 \le \rho_2 \le C'\rho_1$ for some $C, C' > 0$

Remark. intuition: equivalent metrics define the same open, closed, and compact sets, teh same convergent and couchy sequences, and the same continuous and uniformly continuous mappings
so particular metric chosen doesnt matter, only its equivalence class