Contents

1. THE REAL AND COMPLEX NUMBER SYSTEMS

INTRODUCTION.

Assume. integer arithmatic axioms,
$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$

1.1. **Example.** $p^2 = 2 \Rightarrow p \notin \mathbb{Q}$
picture: right triangle with sides $1, 1,$ "$\sqrt{2}$"

1.2. *Remark.* $\mathbb{Q}$ has gaps

1.3. **Definition.** *set theory notation*
- (i) $\in$ means "is an element of set"
- (ii) $\emptyset$ is "empty set"
- (iii) $\subseteq$ means "is a subset of"
  $A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$
- (iv) $\subset$ means "is a proper subset of"
- (v) $=$ means "is the same as"
  $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

1.4. **Definition.** $\mathbb{Q}$ is the set of rational numbers.

ORDERED SETS.

1.5. **Definition.** *order*
order is relation "$<$" on $Set \times Set$ with properties:
(trichotomy) only one of $\{x < y, x = y, x > y\}$ is true
(transitivity) $x < y, y < z \Rightarrow x < z$
note: $x \leq y \Rightarrow x < y$ or $x = y$; negation of $x > y$

1.6. **Definition.** *ordered set*
is a set with order defined.
ex. $\mathbb{Q}$ has order $r < s$ defined by $s - r \in \mathbb{Q}_+$

1.7. **Definition.** *upper bound*
Let S ordered set, $E \subset S$.
$\beta \in S$ is an upper bound of E means $x \leq \beta \ \forall x \in E$
similarly for *lower bound*

1.8. **Definition.** *least upper bound*
Let S ordered set, $E \subset S$, E bounded above.
$\alpha \in S$ is the least upper bound of E means
- (i) α is an upper bound of E
- (ii) $\gamma < \alpha \Rightarrow \gamma$ is not an upper bound of E

Notation: $\sup E = \alpha$
Similarly for *greatest lower bound*, inf

1.9. **Example.**
- (a) $A = \{x \in \mathbb{Q} : x^2 < 2\}$
  A has no l.u.b. in $\mathbb{Q}$
- (b) Let $E_1 = \{r \in \mathbb{Q} : r < 0\}$
  Let $E_2 = \{r \in \mathbb{Q} : r \leq 0\}$
  Then $\sup E_1 = \sup E_2 = 0$
  note: $0 \notin E_1$
- (c) Let $E = \{\frac{1}{n} : n = 1, 2, 3, ...\}$
  Then $\sup E = 1 \in E$
  $\inf E = 0 \notin E$

1.10. **Definition.** *l.u.b. property ("complete")*
Ordered set S has l.u.b.p. means
$\forall E \subset S$ nonempty bounded above, $\sup E \in S$
Similarly for g.l.b.p.

1.11. **Theorem.** *relation between lubp and glbp*
S ordered set with l.u.b.p.
$B \subset S$, $B \neq \emptyset$, B bounded below
L be the set of all lower bounds of B
$\Rightarrow \alpha = \sup L = \inf B$ exists in S
i.e. lubp $\Rightarrow$ glbp.

FIELDS.

1.12. **Definition.** *field, field axioms*
A *field* is a set F with operations $+, \times$ and satisfies the *field axioms*: $\forall x, y, z \in F$,
- (A1) $x + y \in F$
- (A2) $x + y = y + x$
- (A3) $(x + y) + z = x + (y + z)$
- (A4) $\exists 0 \in F$ s.t. $0 + x = x$
- (A5) $\exists(-x) \in F$ s.t. $x + (-x) = 0$
- (M1) $xy \in F$
- (M2) $xy = yx$
- (M3) $(xy)z = x(yz)$
- (M4) $\exists 1 \in F$ s.t. $1x = x$
- (M5) $x \neq 0, \exists \frac{1}{x} \in F$ s.t. $x\frac{1}{x} = 1$
- (D) $x(y + z) = xy + xz$

1.13. *Remark.*
- (a) shorthand notation: $x - y, \frac{x}{y}, x + y + z, xyz, x^3, etc.$
- (b) $\mathbb{Q}$ satisfies field axioms $\Rightarrow \mathbb{Q}$ is a field
- (c) we will prove some field properties using $\mathbb{Q}$ which will hold for $\mathbb{R}, \mathbb{C}$

1.14. **Proposition.** *Axioms of addition $\Rightarrow$*
- *(a)* $x + y = x + z \Rightarrow y = z$
- *(b)* $x + y = x \Rightarrow y = 0$
- *(c)* $x + y = 0 \Rightarrow y = -x$
- *(c)* $-(-x) = x$

1.15. **Proposition.** *Axioms of multiplication $\Rightarrow$*
- *(a)* $x \neq 0, xy = xz \Rightarrow y = z$
- *(b)* $x \neq 0, xy = x \Rightarrow y = 1$
- *(c)* $x \neq 0, xy = 1 \Rightarrow y = \frac{1}{x}$
- *(c)* $x \neq 0 \Rightarrow \frac{1}{\frac{1}{x}} = x$

1.16. **Proposition.** *Field axioms $\Rightarrow$*
- *(a)* $0x = 0$
- *(b)* $x, y \neq 0 \Rightarrow xy \neq 0$
- *(c)* $(-x)y = -(xy) = x(-y)$
- *(c)* $(-x)(-y) = xy$

1.17. **Definition.** *ordered field, positive, negative*
- An *ordered field* is a field and ordered set s.t.
  - (i) $y < z \Rightarrow x + y < x + z$
  - (ii) $x, y > 0 \Rightarrow xy > 0$

  i.e. order preserved by field axioms
  e.g. $\mathbb{Q}$ is an ordered field
- x is *positive* means $x > 0$
- x is *negative* means $x < 0$

1.18. **Proposition.** *Ordered field $\Rightarrow$*
- *(a)* $x > 0 \Rightarrow -x < 0$ *and vice-versa*
- *(b)* $x > 0, y < z \Rightarrow xy < xz$
- *(c)* $x < 0, y < z \Rightarrow xy > xz$
- *(d)* $x \neq 0 \Rightarrow x^2 > 0$, *in particular,* $1 > 0$
- *(e)* $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$

THE REAL FIELD.

1.19. **Theorem.** *existence*
$\exists$ an ordered field $\mathbb{R}$ which has lubp.
Moreover, $\mathbb{Q} \subset \mathbb{R}$

1.20. **Theorem.** *Let $x, y \in \mathbb{R}$*
(archimedean property) $x > 0 \Rightarrow \exists n \in \mathbb{Z}_+$ s.t. $nx > y$
(dense) $x < y \Rightarrow \exists p \in \mathbb{Q}$ s.t. $x < p < y$

1.21. **Theorem.** *$\exists! n^{th}$ root*
$\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}$
$\exists! y > 0$ real s.t. $y^n = x$
notation: $y = x^{\frac{1}{n}} = \sqrt[n]{x}$

Corollary. *Let $a, b \in \mathbb{R}_+$, $n \in \mathbb{N}$,*
$\Rightarrow (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

1.22. **Definition.** *decimals* unused

EXTENDED REAL NUMBER SYSTEM.

1.23. **Definition.** $\bar{\mathbb{R}}$
- $\bar{\mathbb{R}}$ is $\mathbb{R}$ union symbols $+\infty, -\infty$
- preserve order on $\bar{\mathbb{R}}$ by defining
  $-\infty < x < +\infty \ \forall x \in \mathbb{R}$
- every nonempty subset of $\bar{\mathbb{R}}$ has lub
- (arithmatic) $\forall x \in R$,
  - (a) $x + (+\infty) = +\infty$
    $x - (+\infty) = -\infty$
    $\frac{x}{+\infty} = \frac{x}{-\infty} = 0$
  - (b) $x > 0 \Rightarrow x(+\infty) = +\infty$, $x(-\infty) = -\infty$
  - (c) $x < 0 \Rightarrow x(+\infty) = -\infty$, $x(-\infty) = +\infty$

note: not a field since $+\infty + (-\infty), 0 \cdot (+\infty), \frac{+\infty}{+\infty}$ undefined

THE COMPLEX FIELD.

1.24. **Definition.** *complex number, $=, +, \times$*
A complex number is a 2-tuple in $\mathbb{R}$
Let $x, y \in \mathbb{C}$, $x = (a, b)$, $y = (c, d)$
$(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$
$x + y = (a + c, b + d)$
$xy = (ac - bd, ad + bc)$

1.25. **Theorem.** *$\mathbb{C}$ is a field with $+, \times$, zero $(0, 0)$, and one $(1, 0)$*

1.26. **Theorem.** $a, b \in \mathbb{R}$
$\Rightarrow (a, 0) + (b, 0) = (a + b, 0)$, $(a, 0)(b, 0) = (ab, 0)$
note: $\mathbb{R} \subset \mathbb{C}$

1.27. **Definition.** $i = (0, 1)$

1.28. **Theorem.** $i^2 = -1$

1.29. **Theorem.** $a, b \in \mathbb{R} \Rightarrow (a, b) = a + bi$

1.30. **Definition.** *conjugate; real, imaginary part*
Let $a, b \in \mathbb{R}$, $z = a + bi$
the *conjugate* is $\bar{z} = a - bi$
the *real part* $\mathrm{Re}(z) = a$
the *imaginary part* $\mathrm{Im}(z) = b$

1.31. **Theorem.** $z, w \in \mathbb{C} \Rightarrow$
- *(a)* $\overline{z + w} = \bar{z} + \bar{w}$
- *(b)* $\overline{zw} = \bar{z}\bar{w}$
- *(c)* $z + \bar{z} = 2\,\mathrm{Re}(z)$, $z - \bar{z} = 2i\,\mathrm{Im}(z)$
- *(d)* $z \neq 0 \Rightarrow z\bar{z} \in \mathbb{R}_+$

1.32. **Definition.** *absolute value ("modulus")*
Let $z \in \mathbb{C}$
$|z| = (z\bar{z})^{\frac{1}{2}} \geq 0$
note: $\exists!$ from 1.21 and 1.31(d)
note: $x \in \mathbb{R} \Rightarrow x = \bar{x}$, $|x| = \sqrt{x^2}$, etc.

1.33. **Theorem.** $z, w \in \mathbb{C} \Rightarrow$
- *(a)* $|z| > 0$ *unless* $z = 0$
- *(b)* $|\bar{z}| = |z|$
- *(c)* $|zw| = |z||w|$
- *(d)* $Re(z) \leq |z|$
- *(e)* $|z + w| \leq |z| + |w|$

Remark. summation notation
Let $x_1, x_2, ..., x_n \in \mathbb{C}$
Write $x_1 + x_2 + \cdots + x_n = \sum_{j=1}^{n} x_j$

1.34. **Theorem.** *Schwarz inequality*
$a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \in \mathbb{C}$
$\Rightarrow \left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$

EUCLIDEAN SPACES.

1.35. **Definition.** *$\mathbb{R}^k$, vector ("point"), coordinates, vector space over the real field, inner ("scalar", "dot") product, norm, Euclidean k-space*
- $\mathbb{R}^k = \{(x_1, \cdots, x_k) : x_i \in \mathbb{R}\}$
- *vector* $\mathbf{x} \in \mathbb{R}^k$ has *coordinates* $x_1, x_2, ..., x_k$
- addition: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, ..., x_k + y_k) \in \mathbb{R}^k$
- multiplication by scalar $\alpha \in \mathbb{R}$
  $\alpha\mathbf{x} = (\alpha x_1, ..., \alpha x_k) \in \mathbb{R}^k$
- R^k is a *vector space over the real field* means above two operations satisfy commutative, assiciative, distributive, and $\mathbf{0} = (0, ..., 0) \in R^k$
- *inner product* $\mathbf{x} \cdot \mathbf{y}$ is $\sum_{j=1}^{k} x_j y_j = < x, y >$

- *norm* $|\mathbf{x}|$ is $(\mathbf{x}\cdot\mathbf{x})^{\frac{1}{2}}$
- *Euclidean k-space* is $\mathbb{R}^k$ with inner product and norm

1.36. **Theorem.** $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^k$, $\alpha\in\mathbb{R}\Rightarrow$

(a) $|\mathbf{x}|\geq 0$
(b) $|\mathbf{x}|=0\Leftrightarrow\mathbf{x}=\mathbf{0}$
(c) $|\alpha\mathbf{x}|=|\alpha||\mathbf{x}|$
(d) $|\mathbf{x}\cdot\mathbf{y}|\leq|\mathbf{x}||\mathbf{y}|$ *(Schwarz $\leq$)*
(e) $|\mathbf{x}+\mathbf{y}|\leq|\mathbf{x}|+|\mathbf{y}|$
(f) $|\mathbf{x}-\mathbf{z}|\leq|\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{z}|$

1.37. *Remark. metric space*
Thm 1.36(a,b,f) will be used in Ch2 to regard $\mathbb{R}^k$ as a *metric space.*

2. Basic Topology

FINITE, COUNTABLE, AND UNCOUNTABLE SETS.

2.1. **Definition.** *function (mapping, operation), domain, codomain, range*

- *function* f from set A to set B, $f:A\to B$, means $\forall x\in A$, $\exists f(x)\in B$
- f is defined on *domain* A into *codomain* B
- The *range* of f is $f(A)=\{f(x):x\in A\}$.

2.2. **Definition.** *image, onto ("surjection", $\twoheadrightarrow$), inverse image ("preimage"), 1-1 ("injection", $\hookrightarrow$)*
Let A,B sets, $f:A\to B$

- The *image* of $E\subseteq A$ under f is $f(E)=\{f(x):x\in E\}$
- f maps A *onto* B means $f(A)=B$
- The *inverse image* of $E\subseteq B$ under f is $f^{-1}(E)=\{x\in A:f(x)\in E\}$
  The *inverse* of $y\in B$ under f is $f^{-1}(y)=\{x\in A:f(x)=y\}$ (need not be unique)
- f is a $1-1$ mapping means
  $\forall y\in B$, there is at most one $f^{-1}(y)\in A$
  $(\Leftrightarrow x_1\neq x_1\Rightarrow f(x_1)\neq f(x_2))$

2.3. **Definition.** *1-1 correspondence ("bijection", $\hookrightarrow\!\!\!\!\twoheadrightarrow$), equivalence relation*

- sets A,B have *1-1 correspondence*, $A\sim B$, means $\exists$ mapping 1-1 and onto
- Properties of *equivalence relation* $\sim$:
  (reflexive) $A\sim A$
  (symmetric) $A\sim B\Rightarrow B\sim A$
  (transitive) $A\sim B, B\sim C\Rightarrow A\sim C$

2.4. **Definition.** *finite, infinite, countable ("enumerable"), uncountable*
Let $J_n=\{1,2,\cdots,n\}$ $\forall n\in\mathbb{N}$ (can also start with 0 when convenient)

(a) A *finite* means $A\sim J_n$ for some n
(b) A *infinite* means not finite (also see 2.6)
(c) A *countable* means $A\sim\mathbb{N}$
(d) A *uncountable* means A neither finite or countable
(e) A *at most countable* means A finite or countable

2.5. **Example.** $\mathbb{Z}$ is countable
Pf. $f(n)=\begin{cases}\frac{n}{2} & \text{n even}\\ -\frac{n-1}{2} & \text{n odd}\end{cases}$

2.6. *Remark. Alternative definition for infinite set*
Infinite sets can be $\sim$ to proper subsets.
i.e. $B\subset A, B\sim A\Rightarrow A$ infinite

2.7. **Definition.** *sequence, elements ("values", "terms")*

- *sequence* $(x_n)_{n\in\mathbb{N}}\subset A$ is a function $f:\mathbb{N}\to A$, $f(n)=x_n$.
- *elements* are x_n; need not be distinct

e.g. every countable set's elements can be arranged in a sequence.

2.8. **Theorem.** *Set A countable*
$\Rightarrow$ every infinite subset is countable.
note: countable sets represent the smallest infinity. No uncountable set can be a subset of a countable set.

UNIONS AND INTERSECTIONS.

2.9. **Definition.** *set of sets ("collection", "family"), union, intersection, disjoint, intersect*
Let A,Ω sets.

- A *set of sets* is $\{E_\alpha\subset\Omega:\alpha\in A\}$.
- *Union* $S=\bigcup_{\alpha\in A}E_\alpha$ means $x\in S\Leftrightarrow x\in E_\alpha$ for at least one E_α
- *Intersection* is similar.
- *Disjoint* means $A\bigcap B=\emptyset$
- *Intersect* otherwise.

2.10. **Example.**

(a) $E_1=\{1,2,3\}$, $E_2=\{2,3,4\}\Rightarrow$
$E_1\bigcup E_2=\{1,2,3,4\}$ and $E_1\bigcap E_2=\{2,3\}$
(b) Let $A=\{x\in\mathbb{R}:0<x\leq 1\}$
Let $E_x=\{y\in\mathbb{R}:0<y<x\}$ $\forall x\in A$
Then
  (i) $E_x\subset E_z\Leftrightarrow 0<x\leq z\leq 1$
  (ii) $\bigcup_{x\in A}E_x=E_1$
  (iii) $\bigcap_{x\in A}E_x=\emptyset$

2.11. *Remark.* Unions and intersections are similar to sums and products.
(commutative) $A\bigcup B=B\bigcup A$, $A\bigcap B=B\bigcap A$
(associative) $(A\bigcup B)\bigcup C=A\bigcup(B\bigcup C)$, $(A\bigcap B)\bigcap C=A\bigcap(B\bigcap C)$
(distributive) $A\bigcap(B\bigcup C)=(A\bigcap B)\bigcup(A\bigcap C)$
(other relations)
$A\subset A\bigcup B$
$A\bigcap B\subset A$
$A\subset B\Rightarrow A\bigcup B=B$, $A\bigcap B=A$
$\emptyset$ behaves like zero. $A\bigcup\emptyset=A$, $A\bigcap\emptyset=\emptyset$

2.12. **Theorem.** *$(E_n)_{n=1,2,\ldots}$ is a sequence of countable sets.*
$\Rightarrow\bigcup_{n=1}^{\infty}E_n$ is countable

Corollary. *A is at most countable and $\forall\alpha\in A$, B_α is at most countable.*
$\Rightarrow\bigcup_{\alpha\in A}B_\alpha$ is at most countable.

2.13. **Theorem.** *A is countable and $B_n=\{(a_1,a_2,\cdots,a_n):a_k\in A,k=1,\cdots,n\}$ where elements a_k need not be distinct.*
$\Rightarrow B_n$ is countable.

Corollary. *$\mathbb{Q}$ is countable.*

2.14. **Theorem.** *not all infinite sets are countable*
A is the set of all sequences with elements 0 and 1.
$\Rightarrow A$ is uncountable.
Pf: Cantor diagonalization or thm 2.43
e.g. the base 2 representation of $\mathbb{R}$ is uncountable.

METRIC SPACES.

2.15. **Definition.** *metric space, metric ("distance function"), point (element)*
Metric space is a set X with a *metric* $d:X^2\to\mathbb{R}$, any two *points* $p,q\in X$ must satisfy
(nonnegative) $d(p,q)>0$ if $p\neq q$, $d(p,p)=0$
(symmetric) $d(p,q)=d(q,p)$
(triangle $\leq$) $d(q,p)\leq d(q,r)+d(r,p)$ $\forall r\in X$

2.16. **Example.**

- Euclidean space $\mathbb{R}^k$ with $d(\mathbf{x},\mathbf{y})=|\mathbf{x}-\mathbf{y}|$, $\mathbf{x},\mathbf{y}\in\mathbb{R}^k$
- every subset of a metric space is itself a metric space with same distance function

OPEN AND CLOSED SETS.

2.17. **Definition.** *segment, interval, k-cell, ball, convex*
Let $\mathbf{a},\mathbf{b},\mathbf{x},\mathbf{y}\in\mathbb{R}^k$

- For $k=1$
  segment $(a,b)=\{x\in\mathbb{R}:a<x<b\}$
  interval $[a,b]=\{x\in\mathbb{R}:a\leq x\leq b\}$
- For $k\in\mathbb{N}$
  k-cell $[\mathbf{a},\mathbf{b}]=\{\mathbf{x}:a_i\leq x_i\leq b_i\}$
  e.g. 1-cell is interval, 2-cell is rectangle
- *ball* $B_r(\mathbf{x})=\{\mathbf{y}:|\mathbf{x}-\mathbf{y}|<r\}$
  i.e. $\mathbf{x}$=center, r=radius
- Let $E\subset\mathbb{R}^k$, $\mathbf{x},\mathbf{y}\in E$, $0<\lambda<1$
  E is *convex* means $\lambda\mathbf{x}+(1-\lambda)\mathbf{y}\in E$
  e.g. balls and k-cells are convex

2.18. **Definition.** *neighborhood, limit ("accumulation") point, isolated point, closed, interior point, open, complement, perfect, bounded, dense*
Let X metric space, $E\subseteq X$, $p,q\in E$

(a) *neighborhood* of p is $N_r(p)=\{q:d(p,q)<r,r>0\}$
e.g. segments in $\mathbb{R}$, balls in $\mathbb{R}^k$
(b) p is a *limit point* of E means every $N_r(p)$ contains another point $q\in E$
(c) p is *isolated point* of E means p not limit point
(d) E is *closed* means every limit point of E is in E
(e) p is *interior point* of E means $\exists N_r(p)\subset E$
(f) E is *open* means every point is interior point
(g) *complement* $E^c=\{s:s\notin E\}$
(h) E is *perfect* means closed and every $p\in E$ is a limit point of E
(i) E is *bounded* means $\exists M\in\mathbb{R},q\in X$ s.t. $d(p,q)<M$ $\forall p\in E$
(j) E *dense in* X means every $p\in X$ is a limit point of E, a point of E, or both

2.19. **Theorem.** *Every neighborhood is open*

2.20. **Theorem.** *p is a limit point of E*
$\Rightarrow$ every $N_r(p)$ contains infinitely many points of E

Corollary. *A finite set has no limit points*

2.21. **Example.** Consider subsets of $\mathbb{R}^2$ or $\mathbb{R}$

(a) $\{z\in\mathbb{C}:|z|<1\}$
(b) $\{z\in\mathbb{C}:|z|\leq 1\}$
(c) a nonempty finite set
(d) $\mathbb{Z}$
(e) $\{\frac{1}{n}:n=1,2,3,\ldots\}$
note: no point of E is a limit point (e.g. $0\notin E$)
(f) $\mathbb{C}$
(g) (a,b)

	Closed	Open	Perfect	Bounded
(a)	No	Yes	No	Yes
(b)	Yes	No	Yes	Yes
(c)	Yes	No	No	Yes
(d)	Yes	No	No	No
(e)	No	No	No	Yes
(f)	Yes	Yes	Yes	No
(g)	No		No	Yes

note: (g) open depends on if its $\mathbb{R}$ or $\mathbb{R}^2$

2.22. **Theorem.** *DeMorgan*
Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α
$\Rightarrow(\cup_\alpha E_\alpha)^c=\cap_\alpha(E_\alpha^c)$, $(\cap_\alpha E_\alpha)^c=\cup_\alpha(E_\alpha^c)$

2.23. **Theorem.** *set E is open*
$\Leftrightarrow E^c$ is closed

Corollary. *set F is closed*
$\Leftrightarrow F^c$ is open

2.24. **Theorem.**

(a) for any collection $\{G_\alpha\}$ of open sets, $\cup_\alpha G_\alpha$ is open
(b) for any collection $\{F_\alpha\}$ of closed sets, $\cap_\alpha F_\alpha$ is closed

(c) *for any finite collection $G_1, \cdots, G_n$ of open sets, $\cap_{i=1}^n G_i$ is open*
(d) *for any finite collection $F_1, \cdots, F_n$ of closed sets, $\cup_{i=1}^n F_i$ is closed*

2.25. **Example.**
(1) Finiteness essential for (c) and (d) above
(2) $\cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = 0$ so the intersection of an infinite collection of open sets need not be open
(3) $\cup_{n=2}^{\infty}[\frac{1}{n}, 1-\frac{1}{n}] = (0,1)$ so the union of an infinite collection of closed sets need not be closed

2.26. **Definition.** *closure*
The closure of metric space $E \subseteq X$ is $E \cup \{\text{all limit points of } E \text{ in } X\}$
Notation: $\bar{E}, \text{cl}(E)$.

2.27. **Theorem.** *X metric space and $E \subseteq X \Rightarrow$*
(a) *$\bar{E}$ is closed*
(b) *$E = \bar{E} \Leftrightarrow E$ is closed*
(c) *$\bar{E} \subseteq F$ for every closed set $F \subseteq X$ s.t. $E \subseteq F$*

2.28. **Theorem.** *$E \subset \mathbb{R}$ nonempty, bounded above*
$\Rightarrow \sup E \in \bar{E}$
hence: E closed $\Rightarrow \sup E \in E$

2.29. *Remark.* define *open relative to*
Let X metric space, $E \subset Y \subset X$
E open relative to X means ($\forall p \in E, \exists r > 0$ s.t. $d(p,q) < r, q \in X \Rightarrow q \in E$)
Similarly, E can be open relative to Y
recall ex 2.21(g), E can be open relative to Y and not X; similarly for closed relative to.
next: relation between these concepts

2.30. **Theorem.** *Let $E \subset Y \subset X$*
E is open relative to Y
$\Leftrightarrow E = Y \cap G$ for some open $G \subset X$

COMPACT SETS.

2.31. **Definition.** *open cover*
An *open cover* of metric space $E \subset X$ is a collection $\{G_\alpha\}$ of open subsets of X s.t. $E \subset \cup_\alpha G_\alpha$

2.32. **Definition.** *compact*
Metric space $K \subset X$ is *compact* means every open cover of K contains a *finite subcover*
i.e. $K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$
Important connection to continuity in ch.4
e.g. every finite set is compact
$\exists$ a large class of infinite compact sets in $\mathbb{R}^k$ (see 2.41)
Unlike open, closed (remark 2.29), compactness unaffected by the space where set is embedded.

2.33. **Theorem.** *Let $K \subset Y \subset X$ metric spaces*
K is compact relative to X
$\Leftrightarrow K$ is compact relative to Y.
note: now we are able to regard compact sets as metric spaces without paying attention to any embedding space.
it makes little sense to talk of open or closed spaces, since every metric space is both closed and open subset of itself
it makes more sense to talk about compact metric spaces.

2.34. **Theorem.** *Compact subsets of metric spaces are closed.*

2.35. **Theorem.** *Closed subsets of compact sets are compact.*

Corollary. *F closed, K compact*
$\Rightarrow F \cap K$ compact

2.36. **Theorem.** *Finite intersection property*
$\{K_\alpha\}$ a collection of compact subsets of metric space X
every finite subcollection has nonempty intersection
$\Rightarrow \cap_\alpha K_\alpha \neq \emptyset$

Corollary. *Nested compact*
sequence $(K_n)_{n=1,2,\dots}$ of nonempty compact sets s.t. $K_{n+1} \subset K_n$
$\Rightarrow \cap_1^\infty K_n \neq \emptyset$

2.37. **Theorem.** *E an infinite subset of compact set K*
$\Rightarrow E$ has a limit point in K

2.38. **Theorem.** *Nested interval*
Sequence $(I_n)_{n=1,2,\dots}$ of intervals in $\mathbb{R}$ s.t. $I_{n+1} \subset I_n$
$\Rightarrow \cap_1^\infty I_n \neq \emptyset$

2.39. **Theorem.** *Nested k-cell*
Let $k \in \mathbb{N}$
Sequence $(I_n)_{n=1,2,\dots}$ of k-cells s.t. $I_{n+1} \subset I_n$
$\Rightarrow \cap_1^\infty I_n \neq \emptyset$

2.40. **Theorem.** *Every k-cell is compact.*

2.41. **Theorem.** *Heine-Borel*
Let $E \subset \mathbb{R}^k$. All or none are true
(a) *E closed and bounded*
(b) *E compact*
(c) *Every infinite subset of E has a limit point in E*

note: in general metric space, (a)$\not\Rightarrow$(b),(c)

2.42. **Theorem.** *Bolzano-Weierstrass*
Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

PERFECT SETS.

2.43. **Theorem.** *$P \subset \mathbb{R}^k$ nonempty, perfect*
$\Rightarrow P$ uncountable
e.g. $\mathbb{R}$ uncountable

2.44. **Definition.** *cantor set*
Will show perfect set in $\mathbb{R}$ that contains no segment.
Start with interval $E_0 = [0,1]$
remove segment $(\frac{1}{3}, \frac{2}{3})$ to get $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
and again to get $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
continue and see that (a)$E_0 \supset E_1 \supset E_2 \supset \cdots$ and (b)E_n is the union of 2^n intervals each of length 3^{-n}
The *cantor set* is $P = \cap_1^\infty E_n$
P is compact (clearly) and nonempty (2.36).
P contains no segment (see book).
P is perfect since contains no isolated point.
P is uncountable with measure zero (ch 11).

CONNECTED SETS.

2.45. **Definition.** *separated, connected*
Let X metric space.
$A, B \subset X$ are *separated* means both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.
$E \subset X$ is *connected* means E is not a union of two nonempty separated sets.

2.46. *Remark.* Separated sets are disjoint.
Disjoint sets need not be separated e.g. $(0,1)$, $[1,2]$

2.47. **Theorem.** *$E \subset \mathbb{R}$ is connected*
$\Leftrightarrow$ ($x, y \in E$, $x < z < y \Rightarrow z \in E$).

3. NUMERICAL SEQUENCES AND SERIES

CONVERGENT SEQUENCES.

Remark. Will consider sequences in $\mathbb{C}$, $\mathbb{R}^k$, and general metric spaces X

3.1. **Definition.** X *converge in, limit, diverge, range, bounded*
- $(p_n) \subset X$ *converges in* metric space X to *limit* $p \in X$, $\lim_{n\to\infty} p_n = p$, means
  $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(p, p_n) < \epsilon$
- (p_n) *diverges* if it does not converge.
- *range* is the set of points p_n; can be finite or infinite set
- (p_n) *bounded* means its range is bounded

Example. $\mathbb{C}$
(a) $s_n = \frac{1}{n} \Rightarrow s_n \to 0$; range is infinite, sequence is bounded
(b) $s_n = n^2 \Rightarrow (s_n)$ divergent; infinite, unbounded
(c) $s_n = 1 + \frac{(-1)^n}{n} \Rightarrow s_n \to 1$; infinite, bounded
(d) $s_n = i^n \Rightarrow (s_n)$ divergent; finite, bounded
(e) $s_n = 1 \Rightarrow s_n \to 1$; finite, bounded

3.2. **Theorem.** *Let $(p_n) \subset$ X*
(a) *$p_n \to p \in X \Leftrightarrow$ every $N_r(p)$ contains p_n for all but finitely many n.*
(b) *$p, p' \in X$, $p_n \to p$, $p_n \to p' \Leftrightarrow p = p'$.*
(c) *(p_n) converges $\Rightarrow (p_n)$ bounded.*
(d) *p limit point of $E \subset X \Rightarrow \exists (p_n)$ in E s.t. $p_n \to p$*

3.3. **Theorem.** *Algebraic properties of limits*
Let $(s_n), (t_n) \subset$ $\mathbb{C}$, $s_n \to s$, $t_n \to t$.
(a) *$s_n + t_n \to s + t$*
(b) *$cs_n \to cs$, $c + s_n \to c + s$*
(c) *$s_n t_n \to st$*
(d) *$\frac{1}{s_n} \to \frac{1}{s}$ provided $s, s_n \neq 0$*

3.4. **Theorem.** $\mathbb{R}^k$
(a) *$\mathbf{x}_n = (\alpha_{1,n}, \cdots, \alpha_{k,n}) \in \mathbb{R}^k$*
$\mathbf{x}_n \to \mathbf{x} = (\alpha_1, \cdots, \alpha_k)$
$\Leftrightarrow \alpha_{j,n} \to \alpha_j$, $j = 1, \cdots, k$.
(b) *$(\mathbf{x}_n), (\mathbf{y}_n)$ sequences in $\mathbb{R}^k$, (β_n) sequence in $\mathbb{R}$, $\mathbf{x}_n \to x$, $\mathbf{y}_n \to y$, $\beta_n \to \beta$*
$\Rightarrow \mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$, $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{x} \cdot \mathbf{y}$, $\beta_n \mathbf{x}_n \to \beta \mathbf{x}$.

SUBSEQUENCES.

3.5. **Definition.** X *subsequence, subsequential limit*
- *subsequence* of (p_n) is (p_{n_i}) where sequence $(n_i) \in \mathbb{N}$ is ordered $n_1 < n_2 < \cdots$
- A *subsequential limit* is the limit of (p_{n_i}).

note: $p_n \to p \Leftrightarrow$ every $p_{n_k} \to p$

3.6. **Theorem.**
(a) *(p_n) sequence in compact metric space* X
$\Rightarrow$ some (p_{n_k}) converges to a point in X
(b) *(Bolzano-Weierstrass) Every bounded sequence in* $\mathbb{R}^k$ *contains a convergent subsequence.*

3.7. **Theorem.** *The subsequential limits of a sequence $(p_n) \subset$ X form a closed subset of X.*

CAUCHY SEQUENCES.

3.8. **Definition.** *Cauchy sequence*
$(p_n) \subset$ X is *Cauchy sequence* means $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(p_n, p_m) < \epsilon$ if $n, m \geq N$.
note: need not know limit

3.9. **Definition.** *diameter*
The *diameter* of nonempty $E \subseteq$ X, $\text{diam}(E)$, is $\sup\{d(p,q) : p, q \in E\}$
Note: (p_n) is a Cauchy sequence $\Leftrightarrow$ $\text{diam}\{p_N, p_{N+1}, p_{N+2}, \cdots\} \overset{N\to\infty}{\to} 0$

3.10. **Theorem.** *Let metric space* X
(a) *$E \subset X$, $\bar{E}$ closure of E*
$\Rightarrow diam\bar{E} = diamE$
(b) *$(K_n)_{n\in\mathbb{N}}$ sequence of compact sets in X, $K_n \supset K_{n+1}$*
$diamK_n \to 0 \Rightarrow \cap_1^\infty K_n$ consists of exactly one point.

3.11. **Theorem.**
(a) *In any* X
Cauchy sequence $\Rightarrow$ convergent sequence

(b) *In compact* X
Cauchy sequence $\Leftrightarrow$ *converges to point in* X
(c) *"Cauchy criterion" in* $\mathbb{R}^k$
Cauchy sequence $\Leftrightarrow$ *convergent.*

3.12. **Definition.** *complete*
A metric space X is *complete* means every Cauchy sequence converges.
note: thm 3.11 says: all compact metric spaces and all Euclidean spaces are complete.
note: thm 3.11 also implies: every closed subset E of a complete metric space is complete. (ie every Cauchy sequence in E is a Cauchy sequence in X hence it converges to some point $p \in X$, also $p \in E$ since E closed)
e.g. not complete: $\mathbb{Q}$ with $d(x, y) = |x - y|$
note: thm 3.2(c) and def 3.1(d) show that convergent sequence $\Rightarrow$ bounded sequence
with $\Leftrightarrow$ for monotonic sequences in $\mathbb{R}$

3.13. **Definition.** *monotonically increasing, decreasing*
$(s_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is
(a) *monotonically increasing* means $s_n \leq s_{n+1}$
(b) *monotonically decreasing* means $s_n \geq s_{n+1}$

3.14. **Theorem.** *monotonic* $(s_n) \subset \mathbb{R}$ *converges* $\Leftrightarrow (s_n)$ *bounded*

UPPER AND LOWER LIMITS.

3.15. **Definition.** Let $(s_n) \subset \mathbb{R}$.
$s_n \to +\infty$ means $\forall M \in \mathbb{R}\ \exists N$ st $n > N \Rightarrow s_n \geq M$
$s_n \to -\infty$ means $\cdots \Rightarrow s_n \leq M$
note: now symbol $\to$ can be used for some divergent sequences without changing def 3.1 for convergence, limit.

3.16. **Definition.** *upper and lower limits ("limsup" and "liminf")*
Let $(s_n) \subset \mathbb{R}$, $E = \{x \in \mathbb{R} \cup \{-\infty, +\infty\}$: subsequence $s_{n_k} \to x\}$
The *upper limit* of (s_n) is $s^* = \sup E = \limsup_{n\to\infty} s_n$
Similarly for $s_* = \inf E = \liminf_{n\to\infty} s_n$

3.17. **Theorem.** *(s_n), E, and s^* as in def 3.16* $\Rightarrow$
(a) $s^* \in E$
(b) $x > s^* \Rightarrow \exists N \in \mathbb{Z}$ *s.t.* $(n \geq N \Rightarrow s_n < x)$
Moreover, s^* is the only number with properties (a),(b).
Similarly for s_*.

3.18. **Example.**
(a) (s_n) of every $\mathbb{Q}$
$\Rightarrow$ every real is a subsequential limit and $\limsup_{n\to\infty} s_n \to +\infty$, $\liminf_{n\to\infty} s_n \to -\infty$
(b) $(s_n) = \frac{(-1)^n}{1+\frac{1}{n}}$
$\Rightarrow \limsup_{n\to\infty} s_n \to 1$, $\liminf_{n\to\infty} s_n \to -1$
(c) $(s_n) \subset \mathbb{R}$, $s_n \to s$
$\Leftrightarrow \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$

3.19. **Theorem.** *$s_n < t_n$ for $n \geq N$ where N fixed*
$\Rightarrow \liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$, $\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$

SOME SPECIAL SEQUENCES.

3.20. **Example.**
(a) $p > 0 \Rightarrow \lim_{n\to\infty} \frac{1}{n^p} = 0$
(b) $p > 0 \Rightarrow \lim_{n\to\infty} \sqrt[n]{p} = 1$
(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$
(d) $p > 0, \alpha \in \mathbb{R} \Rightarrow \lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$
(e) $|x| < 1 \Rightarrow \lim_{n\to\infty} x^n = 0$

SERIES.

3.21. **Definition.** *sum, partial sum, series ("infinite sum"), converges, diverges*
- A sum is $a_p + a_{p+1} + \cdots + a_q = \sum_{n=p}^q a_n$
- The *partial sum* of (a_n) is (s_n) where $s_n = \sum_{k=1}^n a_k$
- an *infinite series* is $a_1 + a_2 + \cdots = \sum_{n=1}^\infty a_n = \sum a_n$
- The series *converges* means $s_n \to s$ $=\sum_{n=1}^\infty a_n$
- Series *diverges* is complementary

Note: every theorem about sequences can be stated in terms of series with $a_1 = s_1$, $a_n = s_n - s_{n-1}$, and vice versa.

3.22. **Theorem.** *Cauchy criterion (see thm 3.11)*
$\sum a_n$ *converges*
$\Leftrightarrow \forall\epsilon > 0\ \exists N$ *s.t.* $m \geq n \geq N \Rightarrow \left|\sum_{k=n}^m a_k\right| < \epsilon$
e.g. $|a_n| \leq \epsilon$ *for* $m = n$

3.23. **Theorem.** *Cauchy criterion restated*
$\sum a_n$ *converges* $\Rightarrow \lim_{n\to\infty} a_n = 0$
note: $a_n \to 0 \not\Rightarrow \sum a_n$*converges e.g.* $\sum \frac{1}{n}$ *diverges*

3.24. **Theorem.** *restate 3.14 A series of nonnegative ($\mathbb{R}_+$) terms converges*
$\Leftrightarrow$ *its partial sums form a bounded sequence*

3.25. **Theorem.** *comparison test*
(a) $|a_n| \leq c_n$ *for* $n \geq N_0 \in \mathbb{Z}$, $\sum c_n$ *converges*
$\Rightarrow \sum a_n$ *converges*
(b) $a_n \geq d_n \geq 0$ *for* $n \geq N_0$, $\sum d_n$ *diverges*
$\Rightarrow \sum a_n$ *diverges*

note: roots in Cauchy criterion for series i.e. bounded tails

SERIES OF NONNEGATIVE TERMS.

Remark. Need series with which to comparison test

3.26. **Theorem.** *geometric series*
$$\sum_{n=1}^\infty x^n \begin{cases} = \frac{1}{1-x} & 0 \leq x < 1 \\ \textit{diverges} & x \geq 1 \end{cases}$$

3.27. **Theorem.** *Let* $a_1 \geq a_2 \geq \cdots \geq 0$
$\sum_{n=1}^\infty a_n$ *converges*
$\Leftrightarrow \sum_{k=0}^\infty 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \cdots$ *converges*

3.28. **Theorem.**
$$\sum \frac{1}{n^p} \begin{cases} \textit{converges} & p > 1 \\ \textit{diverges} & p \leq 1 \end{cases}$$

3.29. **Theorem.**
$$\sum_{n=2}^\infty \frac{1}{n(\log n)^p} \begin{cases} \textit{converges} & p > 1 \\ \textit{diverges} & p \leq 1 \end{cases}$$

note: log *is logarithm base e*

NUMBER e.

3.30. **Definition.** e
$e = \sum_{n=0}^\infty \frac{1}{n!}$ where $n! = 1 \cdot 2 \cdot 3 \cdots n$, $0! = 1$

3.31. **Theorem.** $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

3.32. **Theorem.** *e is irrational*

THE ROOT AND RATIO TEST.

3.33. **Theorem.** *root test*
Given $\sum a_n$, *put* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. *Then*
$$\sum a_n \begin{cases} \textit{converges} & \alpha < 1 \\ \textit{diverges} & \alpha > 1 \\ \textit{no info} & \alpha = 1 \end{cases}$$

3.34. **Theorem.** *ratio test*
$$\sum a_n \begin{cases} \textit{converges} & \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \\ \textit{diverges} & \left|\frac{a_{n+1}}{a_n}\right| \geq 1\ \forall n \geq n_0 \in \mathbb{Z} \end{cases}$$

3.35. **Example.**
(a) $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$
$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$
$\liminf_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}}$
$\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}}$
$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n = +\infty$
Root test $\Rightarrow$ convergence, Ratio test not applicable.
(b) $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots$
$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}$
$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$
$\lim \sqrt[n]{a_n} = \frac{1}{2}$
Root test $\Rightarrow$ convergence, Ratio test not applicable.

3.36. *Remark.*
- Ratio test is frequently easier to apply than root test since its easier to compute ratios than nth roots
- Root test has wider scope
ratio test convergence $\Rightarrow$ root test convergence
root test inconclusive $\Rightarrow$ ratio test inconclusive
- Both deduce divergence from $a_n \not\to 0$

3.37. **Theorem.** *Let* $(c_n) \subset \mathbb{R}_+$
$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$

POWER SERIES.

3.38. **Definition.** *power series, coefficents, circle of convergence, converge, diverge*
Let $(c_n) \subset \mathbb{C}$ and $z \in \mathbb{C}$
- *power series* is $\sum_{n=0}^\infty c_n z^n$
- *coefficients* of the series are numbers c_n
- A *circle of convergence* associated with every power series, convergence if z in interior of circle, divergence if exterior, can have radius 0 or ∞

3.39. **Theorem.** *Let* $\sum c_n z^n$, $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$
$$\Rightarrow \sum a_n z^n \begin{cases} \textit{converges} & |z| < R \\ \textit{diverges} & |z| > R \end{cases}$$
note: allow R to take values $+\infty, 0$

3.40. **Example.**
(a) $\sum n^n z^n$ has $R = 0$
(b) $\sum \frac{z^n}{n!}$ has $R = +\infty$ (ratio test easier than root test)
(c) $\sum z^n$ has $R = 1$ ($|z| = 1 \Rightarrow$ diverges since $z^n \not\to 0$)
(d) $\sum \frac{z^n}{n}$ has $R = 1$ ($z = 1 \Rightarrow$ diverges, all other $|z| = 1 \Rightarrow$ converges)
(e) $\sum \frac{z^n}{n^2}$ has $R = 1$ ($|z| = 1 \Rightarrow$ converges by comparison test)

SUMMATION BY PARTS.

3.41. **Theorem.** *partial summation formula*
Let $(a_n),(b_n)$, *put* $A_n = \sum_{k=0}^n a_k$, *if* $n \geq 1$, *put* $A_{-1} = 0$, $0 \leq p \leq q$
$\Rightarrow \sum_{n=1}^q a_n b_n = \sum_{n=0}^q (A_n - A_{n-1}) b_n = \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$
note: like integration by parts
picture: rectangles

3.42. **Theorem.** *Let*

- (a) *the partial sums* A_n *of* $\sum a_n$ *form a bounded sequence*
- (b) $b_0 \geq b_1 \geq \cdots$ *(i.e. monotonic)*
- (c) $b_n \to 0$

$\Rightarrow \sum a_n b_n$ *converges*

3.43. **Theorem.** *Alternating series test*
Let

- (a) $|c_1| \geq |c_2| \geq \cdots$
- (b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$, $(m = 1, 2, \cdots)$
- (c) $c_n \to 0$

$\Rightarrow \sum c_n$ *converges*

3.44. **Theorem.** $\sum c_n z^n$, $R = 1$, $c_0 \geq c_1 \geq \cdots$, $c_n \to 0$
$\Rightarrow \sum c_n z^n$ *converges at every point on circle* $|z| = 1$ *except possibly* $z = 1$

ABSOLUTE CONVERGENCE.

Remark. How about when some terms are negative?

3.45. **Theorem.** $\sum a_n$ *converges absolutely* $\Rightarrow \sum a_n$ *converges*
note: converges absolutely means $\sum |a_n|$ *converges*

3.46. *Remark. non-absolute (conditional) convergence*

- for series of pos. terms, abs. conv. $\Leftrightarrow$ conv.
- a_n converges *non-absolutely* (conditionally) means $\sum a_n$ converges, $\sum |a_n|$ diverges
  e.g. $\sum \frac{(-1)^n}{n}$
- comparison, root, and ratio tests are for abs. conv, and not useful for non-abs. conv. series
- summation by parts sometimes useful for non-abs. conv. series
  in particular, power series abs. conv. in interior of circle of conv.
- abs. conv. series can be multiplied term by term, and change order of addition, much like with finite series.
  more care required for non-abs. conv. series

ADDITION AND MULTIPLICATION OF SERIES.

3.47. **Theorem.** $\sum a_n = A$, $\sum b_n = B$, *c fixed*
$\Rightarrow \sum (a_n + b_n) = A + B$ *and* $\sum c a_n = cA$

3.48. **Definition.**

- Let $\sum a_n, \sum b_n$, put $c_n = \sum_{k=0}^n a_k b_{n-k}$ $n = 0, 1, 2, \cdots$
  $\sum c_n$ is the *product* of $\sum a_n$ and $\sum b_n$
- Multiply power series
  $\sum_{n=0}^\infty a_n z^n \cdot \sum_{n=0}^\infty b_n z^n = (a_0 + a_1 z + \cdots) \cdot (b_0 + b_1 z + \cdots)$
  $= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + \cdots = c_0 + c_1 z + \cdots$
  Set $z = 1$ to get above.

3.49. **Example.** Let $A_n = \sum_{k=0}^n a_k$, $B_n = \sum_{k=0}^n b_k$, $C_n = \sum_{k=0}^n c_k$, and $A_n \to A$, $B_n \to B$
Does $C_n \to AB$?
We do not have $C_n = A_n B_n$. The product of two convergent series can actually diverge e.g. conv. (but non-abs. conv.) series $\sum_{n=0}^\infty \frac{(-1)^n}{\sqrt{n+1}}$ times itself diverges

3.50. **Theorem.**

- (a) $\sum_{n=0}^\infty a_n$ *conv. abs.*
- (b) $\sum_{n=0}^\infty a_n = A$
- (c) $\sum_{n=0}^\infty b_n = B$
- (d) $c_n = \sum_{k=0}^\infty a_k b_{n-k}$, $n = 0, 1, 2, \cdots$

$\Rightarrow \sum_{n=0}^\infty c_n = AB$
i.e at least one series must be abs. conv.

3.51. **Theorem.** $\sum a_n, \sum b_n, \sum c_n$ *converge to* A, B, C, *and* $c_n = a_0 b_n + \cdots + a_n b_0$
$\Rightarrow C = AB$

REARANGEMENTS.

3.52. **Definition.** *rearrangement*

- Let $(k_n)_{n=1,2,3,\ldots}$ where every positive integer appears once and only once (i.e. k is 1-1 from $\mathbb{N}$ onto $\mathbb{N}$).
  Put $a'_n = a_{k_n}$, $n = 1, 2, 3, \cdots$.
  $\sum a'_n$ is called a *rearrangement* of $\sum a_n$
- Let $(s_n), (s'_n)$ be sequences of partial sums of $\sum a_n, \sum a'_n$.
  In general, these two sequences consist of different numbers. Want to know under what conditions all rearrangements of a convergent series will converge, and whether the sums are necessarily the same.

3.53. **Example.** Consider convergent series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$
a rearrangement $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \ldots \ldots$
Do not converge to same number.

3.54. **Theorem.** $\sum a_n$ *series of reals converges but not absolutely,* $-\infty \leq \alpha \leq \beta \leq \infty$
$\Rightarrow$ *exists rearrangement* $\sum a'_n$ *with partial sums* s'_n *s.t.*
$\liminf_{n\to\infty} s'_n = \alpha$, $\limsup_{n\to\infty} s'_n = \beta$

3.55. **Theorem.** $\sum a_n$ *series of complex numbers that converges absolutely*
$\Rightarrow$ *every rearrangement of* $\sum a_n$ *converges, and they all converge to the same sum.*

4. CONTINUITY

Remark. Will prove in general context of metric space X, Y.

LIMITS OF FUNCTIONS.

4.1. **Definition.** limit
$f : E \subseteq X \to Y$, $p \in X$ limit point of E, $q \in Y$.
$\lim_{x\to p} f(x) = q$ means
$\forall \epsilon > 0\ \exists \delta > 0$ s.t. $d_Y(f(x), q) < \epsilon\ \forall x \in E$ for which $0 < d_X(x, p) < \delta$
note: for $p \in E$, can have $\lim_{x\to p} f(x) \neq f(p)$

4.2. **Theorem.** *characterize limit with sequences*
reuse notation above
$\lim_{x\to p} f(x) = q$
$\Leftrightarrow \lim_{n\to\infty} f(p_n) = q\ \forall (p_n) \subset E$ *s.t.* $p_n \neq p$, $p_n \to p$

Corollary. *f has limit at p*
$\Rightarrow$ *this limit is unique*

4.3. **Definition.** *addition and multiplication, constant function, function* $\geq$
Let $f, g : E \subseteq X \to \mathbb{C}$

- $f + g, f - g, fg, \frac{f}{g} : E \to \mathbb{C}$ where denominator $g \neq 0$
  note: for $f, g : E \to \mathbb{R}^k$, use dot product
- f is *constant* means $f(x) = c\ \forall x \in E$
- for $x, y \in \mathbb{R}$, $f \geq g$ means $f(x) \geq g(x)$ $\forall x \in E$

4.4. **Theorem.** *Algebraic properties*
$p \in X$ *limit point of* $E \subseteq X$
$f, g : E \to \mathbb{C}$, $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B \Rightarrow$

- (a) $\lim_{x\to p} (f + g)(x) = A + B$
- (b) $\lim_{x\to p} (fg)(x) = AB$
- (c) $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ $B \neq 0$

note: $f, g : E \to \mathbb{R}^k$ *then (b) uses dot product.*

CONTINUOUS FUNCTIONS.

4.5. **Definition.** *continuous, continuous on*

- $f : E \subseteq X \to Y$ is *continuous* at $p \in E$ means
  $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $d_Y(f(x), f(p)) < \epsilon$ $\forall x \in E$ for which $d_X(x, p) < \delta$
- f is *continuous on* E means continuous $\forall p \in E$

note: p must be in E
note: f automatically continuous at isolated point

4.6. **Theorem.** *case:* $p \in E$ *is limit point*
$f : E \subseteq X \to Y$ *is continuous at limit point* $p \in E$
$\Leftrightarrow \lim_{x\to p} f(x) = f(p)$

4.7. **Theorem.** *composition and continuity*
X, Y, Z *metric spaces*
$f : E \subseteq X \to Y$, $g : f(E) \to Z$, $h : E \to Z$,
$h(x) = g(f(x)) = g \circ f(x)\ \forall x \in E$
f continuous at $p \in E$, *g continuous at* $f(p)$
$\Rightarrow$ *h continuous at p*

4.8. **Theorem.** *topological characterization of continuity*
$f : X \to Y$ *continuous on X*
$\Leftrightarrow f^{-1}(V)$ *is open in X* $\forall$*open* $V \subseteq Y$

Corollary. *f continuous*
$\Leftrightarrow f^{-1}(C)$ *is closed in X* $\forall$*closed* $C \subseteq Y$

Remark. next consider $\mathbb{C}$, $\mathbb{R}$, and vector valued functions

4.9. **Theorem.** *Algebraic properties*
$f, g : X \to \mathbb{C}$ *continuous*
$\Rightarrow f + g,\ fg,\ \frac{f}{g}$ *continuous on X for denominator* $\neq 0$

Remark. "components" of $f : X \to \mathbb{R}^k$ are $f_1, ..., f_k$

4.10. **Theorem.** $\mathbb{R}^k$
$f, g : X \to \mathbb{R}^k$

- (a) *f continuous*
  $\Leftrightarrow$ *each component* $f_1, \cdots, f_k$ *continuous*
- (b) *f, g continuous*
  $\Rightarrow f + g, f \cdot g$ *continuous on X*

Remark. define coordinate function
A *coordinate function* is $\phi_i : \mathbb{R}^k \to \mathbb{R}$, $\phi_i(\mathbf{x}) = x_i$ where $\mathbf{x} = (x_1, \cdots, x_k)$

4.11. **Example.** *continuous functions*

- coordinate functions are continuous
- every monomial $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ where $n_1, \cdots, n_k \in \mathbb{N}$ is continuous on $\mathbb{R}^k$
- constant multiples are continuous since constant is continuous
- every polynomial $P(x) = \sum_1^m c_{n_1\ldots n_k} x_1^{n_1} \cdots x_k^{n_k}$, where $c_{n_1\ldots n_k} \in \mathbb{C}$, is continuous on $\mathbb{R}^k$
- every quotient of polynomials is continuous whenever denominator $\neq 0$
- $g(x) = |x|$ is continuous
- Let $f : X \to \mathbb{R}^k$ continuous.
  $\phi(p) = |f(p)|$ is continuous on X

4.12. *Remark.* for continuity of $f : E \subseteq X \to Y$, can ignore $\subseteq X$ since X unimportant. Starting theorem 4.8, ignoring $\subseteq X$

CONTINUITY AND COMPACTNESS.

4.13. **Definition.** *bounded*
$f : E \to \mathbb{R}^k$ is bounded means $\exists M \in \mathbb{R}$ s.t. $|f(x)| \le M \; \forall x \in E$

4.14. **Theorem.** *continuity preserves compactness*
X compact metric space, Y metric space
$f : X \to Y$ continuous
$\Rightarrow f(X)$ is compact

Remark. what are consequences of theorem 4.14

4.15. **Theorem.** *X compact metric space*
$f : X \to \mathbb{R}^k$ continuous
$\Rightarrow f(X)$ is closed, bounded
Thus f is bounded

4.16. **Theorem.** *extrema*
X compact metric space, $f : X \to \mathbb{R}$ continuous, $m = \inf_{p\in X} f(p)$, $M = \sup_{p\in X} f(p)$
$\Rightarrow \exists p, q \in X$ s.t. $f(q) = M, f(p) = m$
i.e. $\exists p, q \in X$ s.t. $f(q) \le f(x) \le f(p) \; \forall x \in \mathbb{R}$
i.e. f attains its extrema

4.17. **Theorem.** *X compact metric space, Y metric space*
$f : X \to Y$ continuous, bijection
$\Rightarrow f^{-1} : Y \to X$ defined $f^{-1}(f(x)) = x$ is continuous

UNIFORMLY CONTINUOUS.

4.18. **Definition.** *uniformly continuous*
$f : X \to Y$ is *uniformly continuous* means
$\forall \epsilon > 0 \; \exists \delta > 0$ s.t. $d_Y(f(p), f(q)) < \epsilon \; \forall p, q \in X$ for which $d_X(p, q) < \delta$
note: uniform continuity is a property of a function on a set whereas continuity is defined at a single point

4.19. **Theorem.** *X compact*
$f : X \to Y$ continuous
$\Rightarrow f$ uniformly continuous on X

4.20. **Theorem.** *compactness essential for thms 4.14,15,16,19*
$E \subseteq \mathbb{R}$ not compact $\Rightarrow$

(a) *$\exists$ continuous function on E which is not bounded*
(b) *$\exists$ continuous and bounded function on E with no maximum*
(c) *E bounded $\Rightarrow \exists$ continuous function on E which is not uniformly continuous*

4.21. **Example.** *compacness essential for theorem 4.17*
$f : [0, 2\pi) \to \{x, y : x^2+y^2 = 1\}$, $f(t) = (\cos t, \sin t)$
note: sin and cos are continuous, periodic (ch 8)$\Rightarrow$ continuous, 1-1
f^{-1} (exists since bijection) fails to be continuous at $f(0) = (1, 0)$

CONTINUITY AND CONNECTEDNESS.

4.22. **Theorem.** *$f : X \to Y$ continuous, $E \subseteq X$ connected*
$\Rightarrow f(E)$ connected

4.23. **Theorem.** *intermediate value theorem*
$f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ continuous, $f(a) < c < f(b)$
$\Rightarrow \exists x \in (a, b)$ s.t. $f(x) = c$
Similarly for $f(a) > c > f(b)$

4.24. *Remark.* converse of theorem 4.23 doesn't hold (see ex 4.27d)

DISCONTINUITIES.

4.25. **Definition.** *left, right-hand limit*

- *right-hand limit* $f(x+)$ is $\lim_{n\to\infty} f(t_n)$ $\forall (t_n) \subset (x, b)$ s.t. $t_n \to x$
- *left-hand limit* is similar

note: $\lim_{t\to x} f(t)$ exists $\Leftrightarrow f(x+) = f(x-) = \lim_{t\to x} f(t)$

4.26. **Definition.** *discontinuity of first kind (simple), second kind*

- $f : (a, b) \to \mathbb{R}$ has *discontinuity of first kind* at x means f discontinuous at x and $f(x+), f(x-)$ exist
  note: 2 possibilities: (1) $f(x+) \ne f(x-)$ and (2) $f(x+) = f(x-) \ne f(x)$
- f has *discontinuity of second kind* otherwise

4.27. **Example.**

(a) $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{I} \end{cases}$
has discontinuity of second kind at each x

(b) $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{I} \end{cases}$
is continuous at $x = 0$, otherwise has discontinuity of second kind

(c) $f(x) = \begin{cases} x+2 & -3 < x < -2 \\ -x-2 & -2 \le x < 0 \\ x+2 & 0 \le x < 1 \end{cases}$
has simple discontinty at $x = 0$, otherwise continuous

(d) $f(x) = \begin{cases} \sin \frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$
has discontinty of second kind at at $x = 0$, otherwise continuous (assuming sin continuous)

MONOTONIC FUNCTIONS.

4.28. **Definition.** *monotonically increasing, decreasing*

- $f : (a, b) \to \mathbb{R}$ *monotonically increasing* means $a < x < y < b \Rightarrow f(x) \le f(y)$
- f *monotonically decreasing* is similar

4.29. **Theorem.** *f monotonically decresing on (a, b)*
$\Rightarrow f(x+)$ and $f(x-)$ exist $\forall x \in (a, b)$
i.e. $\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t)$
furthermore, $a < x < y < b \Rightarrow f(x+) \le f(y-)$
monotonically increasing is similar

Corollary. *Monotonic functions have no discontinuities of the second kind*

4.30. **Theorem.** *The set of points at which monotonic $f : (a, b) \to \mathbb{R}$ is discontinuous is at most countable*

4.31. *Remark.*

- The discontinuities of a monotonic function need not be isolated, can even be dense. See book for example.
- *continuous from the left* means $f(x-) = f(x)$. Similar for right.
- see theorem 6.16 for another method to define functions of this sort

INFINITE LIMITS AND LIMITS AT INFINITY.

Remark. How does above generalize to extended real line?

4.32. **Definition.** *neighborhood of $+\infty$, $-\infty$*
The *neighborhood of* $+\infty$, $(c, +\infty)$, is $\{x : x > c\}$
Similarly for $-\infty$

4.33. **Definition.** *characterize limit to $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ using neighborhoods*
Let $f : E \subseteq \mathbb{R} \to \mathbb{R}$.
$f(t) \to A \in \bar{\mathbb{R}}$ as $t \to x \in \bar{\mathbb{R}}$ means $\forall \; B_\epsilon(A)$, $\exists \; B_\delta(x)$ s.t. $B_\delta(x) \cap E$ nonempty and $f(t) \in B_\epsilon(A)$ $\forall t \in B_\delta(x) \cap E$, $t \ne x$

4.34. **Theorem.** *rewrite theorem 4.4*
$f, g : E \subseteq \mathbb{R} \to \mathbb{R}$
$f(t) \overset{t\to x}{\to} A$, $g(t) \overset{t\to x}{\to} B \Rightarrow$

(a) $f(t) \to A' \Rightarrow A' = A$
(b) $(f+g)(t) \overset{t\to x}{\to} A+B$
(c) $(fg)(t) \overset{t\to x}{\to} AB$
(d) $\frac{f}{g}(t) \overset{t\to x}{\to} \frac{A}{B}$, $B \ne 0$

5. DIFFERENTIATION

Remark.

- Will consider real valued functions on intervals or segments
- In last section, vector valued functions on intervals or segments
- functions from $\mathbb{R}^k$ in chapter 9

DERIVATIVE OF A REAL VALUED FUNCTION.

5.1. **Definition.** *derivative, derivative on, right,left-hand derivatives*

- The *derivative* of $f : [a, b] \to \mathbb{R}$ at $x \in [a, b]$ is $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$, where $t \in (a, b)$, $t \ne x$, if it exists
- f is *differentiable on* set $E \subseteq [a, b]$ means $f'(x)$ exists $\forall x \in E$
- The *left,right-hand derivatives* use left,right-hand limits (def 4.25); useful at endpoints

5.2. **Theorem.** *$f : [a, b] \to \mathbb{R}$ differentiable at $x \in [a, b]$*
$\Rightarrow f$ continuous at x

Remark. In ch.9, will see converse not true, in fact, $\exists$ functions continuous everywhere, differentiable nowhere

5.3. **Theorem.** *Algebraic properties; sum, product, quotient rules*
$f, g : [a, b] \to \mathbb{R}$ differentiable at $x \in [a, b]$
$\Rightarrow f + g, fg, \frac{f}{g}$ differentaible at x and

(a) $(f + g)'(x) = f'(x) + g'(x)$
(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
(c) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$ *for $g \ne 0$*

5.4. **Example.**

- derivative of constant is zero
- $f(x) = x \Rightarrow f'(x) = 1$
- $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ for $n \in \mathbb{Z}$ and $n < 0 \Rightarrow x \ne 0$
- every polynomial is differentiable
- every rational function (ratio of polynomials) is differentiable except when denominator is zero

5.5. **Theorem.** *chain rule*
$f : [a, b] \to \mathbb{R}$ continuous, f' exists at some $x \in [a, b]$
$g : f([a, b]) \to \mathbb{R}$, g' exists at point $f(x)$
$h : [a, b] \to \mathbb{R}$, $h(t) = g(f(t))$, $t \in [a, b]$
$\Rightarrow h'(x) = g'(f(x))f'(x)$

5.6. **Example.** assume $\sin' x = \cos x$ from ch. 8

(a) $f(x) = \begin{cases} x \sin \frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$
$f'(x) = \begin{cases} \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} & x \ne 0 \\ DNE & x = 0 \end{cases}$

(b) $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$
$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$
note: f' not continuous at $x = 0$

MEAN VALUE THEOREMS.

5.7. **Definition.** *local maxima, minima*
Let X metric space

- $f : X \to \mathbb{R}$ has *local maxima* at $p \in X$ means $\exists \delta > 0$ s.t. $f(q) \leq f(p)\ \forall q \in X$ with $d(p,q) < \delta$
- *local minima* likewise

5.8. **Theorem.** *$f : [a,b] \to \mathbb{R}$ has a local max at $x \in (a,b)$, $f'(x)$ exists*
$\Rightarrow f'(x) = 0$
Local min analogous

5.9. **Theorem.** *Generalized MVT*
$f, g : [a,b] \to \mathbb{R}$ continuous, differentiable on (a,b)
$\Rightarrow \exists x \in (a,b)$ at which $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$

5.10. **Theorem.** *MVT*
$f : [a,b] \to \mathbb{R}$ continuous, differentiable on (a,b)
$\Rightarrow \exists x \in (a,b)$ at which $f(b) - f(a) = (b-a)f'(x)$

5.11. **Theorem.** *$f : (a,b) \to \mathbb{R}$ differentiable*

(a) *$f'(x) \geq 0\ \forall x \in (a,b) \Rightarrow f$ monotonically increasing*
(b) *$f'(x) = 0\ \forall x \in (a,b) \Rightarrow f$ constant*
(c) *$f'(x) \leq 0\ \forall x \in (a,b) \Rightarrow f$ monotonically decreasing*

THE CONTINUITY OF DERIVATIVES.

Remark. derivatives need not be continuous (ex 5.6b)
derivatives have intermediate value property just like continuous functions on interval (thm 4.23)

5.12. **Theorem.** *Intermediate Value Theorem*
$f : [a,b] \to \mathbb{R}$ differentiable on $[a,b]$, $f'(a) < \lambda < f'(b)$
$\Rightarrow \exists x \in (a,b)$ s.t. $f'(x) = \lambda$
similarly for $>$
similar to thm 4.23

Corollary. *f differentiable on $[a,b]$*
$\Rightarrow f'$ cannot have any simple discontinuities on $[a,b]$

L'HOSPITAL'S RULE.

Remark. useful for evaluating limits

5.13. **Theorem.** *L'Hospital's*
$f, g : [a,b] \to \mathbb{R}$ differentiable on (a,b), $g'(x) \neq 0$ $\forall x \in (a,b)$
use extended $\mathbb{R}$ (def 4.33) ie $-\infty \leq a < b \leq +\infty$
$\frac{f'(x)}{g'(x)} \overset{x \to a}{\to} A$
($f(x) \overset{x \to a}{\to} 0$ and $g(x) \overset{x \to a}{\to} 0$) OR ($g(x) \overset{x \to a}{\to} +\infty$)
$\Rightarrow \frac{f(x)}{g(x)} \overset{x \to a}{\to} A$
Similar for $x \to b$ or $g(x) \to -\infty$

DERIVATIVES OF HIGHER ORDER.

5.14. **Definition.** n^{th} *derivative*
The n^{th} *derivative* of f at x, $f^{(n)}(x)$, is the derivative of $f^{(n-1)}$ if exists

Remark. f^n exists at x
$\Rightarrow f^{(n-1)}$ exists in neighborhood of x and differentiable at x
$\Rightarrow f^{(n-2)}$ differentiable in neighborhood of x, etc.

TAYLOR'S THEOREM.

5.15. **Theorem.** *Taylor's Theorem*
$f : [a,b] \to \mathbb{R}$, $n \in \mathbb{Z}_+$, $f^{(n-1)}$ continuous on $[a,b]$, $f^{(n)}$ exists on (a,b), $\alpha, \beta \in [a,b], \alpha \neq \beta$,
$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$
$\Rightarrow \exists x \in (a,b)$ s.t. $f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$
note: $n = 1 \Rightarrow MVT$
intuition: f can be approximated by a polynomial of degree $n-1$ with error related to $\left|f^{(n)}(x)\right|$

DIFFERENTIATION OF VECTOR VALUED FUNCTIONS.

5.16. *Remark.*

- (complex valued functions)
  $f : [a,b] \to \mathbb{C}$ can use def 5.1, thm 5.2,3
  $f'(x) = (\operatorname{Re} f)' + i(\operatorname{Im} f)'$
  f is differentiable at $x \Leftrightarrow \operatorname{Re} f$ and $\operatorname{Im} f$ differentaible at x
- (vector valued functions)
  $f : [a,b] \subset \mathbb{R} \to \mathbb{R}^k$
  $f' : [a,b] \subset \mathbb{R} \to \mathbb{R}^k$, if exists, satisfies $\lim_{t \to x} \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = 0$ where $|\cdot|$ is norm
  $f' = (f_1', \cdots, f_k')$
  f differentiable at $x \Leftrightarrow$ each $f_1, \cdots, f_n$ differentiable at x
  Can use thms 5.2 and 5.3 with inner product
- can't use MVT or L'Hospital's rule for $\mathbb{C}, \mathbb{R}^k$

5.17. **Example.** *MVT fails*
$f : \mathbb{R} \to \mathbb{C}$, $f(x) = e^{ix} = \cos x + i \sin x$ (from ch 8)
$f(2\pi) - f(0) = 1 - 1 = 0$
but $f'(x) = ie^{ix}$ so $|f'(x)| = 1\ \forall x \in \mathbb{R}$

5.18. **Example.** *L'Hospital's rule fails*
$f, g : (0,1) \to \mathbb{C}$, $f(x) = x$, $g(x) = x + x^2 e^{\frac{i}{x^2}}$
$\frac{f(x)}{g(x)} \overset{x \to 0}{\to} 1$ since $\left| e^{\frac{i}{x^2}} \right| = 1$
$\frac{f'(x)}{g'(x)} \overset{x \to 0}{\to} 0$ since $|g'(x)| = \left| 1 + \left(2x - \frac{2i}{x}\right) e^{\frac{i}{x^2}} \right| \geq \left|2x - \frac{2i}{x}\right| - 1 \geq \frac{2}{x} - 1$ and $\left| \frac{f'(x)}{g'(x)} \right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x}$
So L'Hospital's rule fails

5.19. **Theorem.** *Version of MVT for vector valued functions*
$f : [a,b] \to \mathbb{R}^k$ continuous, f differentiable on (a,b)
$\Rightarrow \exists x \in (a,b)$ s.t. $|f(b) - f(a)| \leq (b-a)\,|f'(x)|$

6. THE RIEMANN-STIELTJES INTEGRAL

Remark.

- Will define Riemann integral using order structure of $\mathbb{R}$
- First real valued functions on intervals, then complex and vector valued functions on intervals
- see ch. 11 for integration over sets

DEFINITION AND EXISTENCE OF THE INTEGRAL.

6.1. **Definition.** *partition, upper and lower Riemann sums, upper and lower Riemann integrals, Riemann integrable.*

- *partition* P of interval $[a,b]$ is $\{x_0, \cdots, x_n\}$ where $a = x_0 \leq \cdots \leq x_n = b$
- The *upper Riemann sum* of bounded $f : [a,b] \to \mathbb{R}$ over $[a,b]$ is $U(P,f) = \sum_{i=1}^n \sup_{x_{i-1} < x < x_i} f(x)(x_i - x_{i-1})$
- The *lower Riemann sum* of bounded $f : [a,b] \to \mathbb{R}$ over $[a,b]$ is $L(P,f) = \sum_{i=1}^n \inf_{x_{i-1} < x < x_i} f(x)(x_i - x_{i-1})$
- The *upper Riemann integral* of f over $[a,b]$ is $\overline{\int_a^b} f dx = \inf_P U(P,f)$
- The *lower Riemann integral* of f over $[a,b]$ is $\underline{\int_a^b} f dx = \sup_P L(P,f)$
- f is *Riemann integrable* means upper and lower Riemann integrals are equal
  notation: $f \in \mathscr{R}$, $\int_a^b f(x)\,\mathrm{d}x$

Remark. For f bounded, upper and lower Riemann integrals exist. The question of their equality is more delicate.

Remark. will generalize definitions before developing theory

6.2. **Definition.** *upper and lower Stieltjes sums, upper and lower Stieltjes integrals, Stieltjes integrable*
Let $\alpha : [a,b] \to \mathbb{R}$ monotonically increasing, bounded

- The *upper Stieltjes sum* of bounded $f : [a,b] \to \mathbb{R}$ wrt α over $[a,b]$ is $U(P,f,\alpha) = \sum_{i=1}^n \sup_{x_{i-1} < x < x_i} f(x)(\alpha(x_i) - \alpha(x_{i-1}))$
- The *lower Stieltjes sum* of bounded $f : [a,b] \to \mathbb{R}$ wrt α over $[a,b]$ is $L(P,f,\alpha) = \sum_{i=1}^n \inf_{x_{i-1} < x < x_i} f(x)(\alpha(x_i) - \alpha(x_{i-1}))$
- The *upper Stieltjes integral* of f wrt α over $[a,b]$ is $\overline{\int_a^b} f\,\mathrm{d}\alpha = \inf_P U(P,f,\alpha)$
- The *lower Stieltjes integral* of f wrt α over $[a,b]$ is $\underline{\int_a^b} f\,\mathrm{d}\alpha = \sup_P L(P,f,\alpha)$
- f is Stieltjes integrable means upper and lower Stieltjes integrals are equal
  notation: $f \in \mathscr{R}(\alpha)$, $\int_a^b f(x)\,\mathrm{d}\alpha(x)$

6.3. **Definition.** *refinement, common refinement*
P^* is a *refinement* of P means $P^* \supset P$
P^* is the common refinement of P_1 and P_2 means $P^* = P_1 \cup P_2$

6.4. **Theorem.** *P^* refines P*
$\Rightarrow L(P,f,\alpha) \leq L(P^*,f,\alpha)$, $U(P^*,f,\alpha) \leq U(P,f,\alpha)$

6.5. **Theorem.** $\underline{\int_a^b} f\,\mathrm{d}\alpha \leq \overline{\int_a^b} f\,\mathrm{d}\alpha$

6.6. **Theorem.** *criterion for integrability*
$f \in \mathscr{R}(\alpha)$ on $[a,b]$
$\Leftrightarrow \forall \epsilon\ \exists P_\epsilon$ s.t. $U(P_\epsilon,f,\alpha) - L(P_\epsilon,f,\alpha) < \epsilon$

6.7. **Theorem.**

(a) *P_ϵ satisfies thm 6.6*
$\Rightarrow$ every refinement does too
(b) *$P_\epsilon = \{x_0, \cdots, x_n\}$ satisfies thm 6.6*
$s_i, t_i \in [x_{i-1}, x_i]$ for all $i \Rightarrow$
$\sum_{i=1}^n |f(s_i) - f(t_i)|\,(\alpha(x_i) - \alpha(x_{i-1})) < \epsilon$
(c) *$f \in \mathscr{R}(\alpha)$ and (b) holds $\Rightarrow$*
$\left|\sum_{i=1}^n f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) - \int_a^b f\,\mathrm{d}\alpha\right| < \epsilon$

6.8. **Theorem.** *f continuous on $[a,b]$*
$\Rightarrow f \in \mathscr{R}(\alpha)$ on $[a,b]$

6.9. **Theorem.** *f monotone on $[a,b]$, α continuous on [a,b] and still monotone*
$\Rightarrow f \in \mathscr{R}(\alpha)$

6.10. **Theorem.** *f bounded on $[a,b]$, f has only finitely many points of discontinuity*
α continuous on each point where f discontinuous
$\Rightarrow f \in \mathscr{R}(\alpha)$

6.11. **Theorem.** *$f \in \mathscr{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$,*
$\phi : [m,M] \to \mathbb{R}$ continuous,
$h : [a,b] \to \mathbb{R}$, $h(x) = \phi(f(x))$
$\Rightarrow h \in \mathscr{R}(\alpha)$ on $[a,b]$

PROPERTIES OF THE INTEGRAL.

6.12. **Theorem.**

(a) *$f_1, f_2 \in \mathscr{R}(\alpha)$ on $[a,b]$, c constant*
$\Rightarrow f_1 + f_2, cf_1 \in \mathscr{R}(\alpha)$
and $\int_a^b (cf_1 + f_2)d\alpha = c\int_a^b f_1\,\mathrm{d}\alpha + \int_a^b f_2\,\mathrm{d}\alpha$
(b) *$f_1(x) \leq f_2(x)$ on $[a,b]$*
$\Rightarrow \int_a^b f_1\,\mathrm{d}\alpha \leq \int_a^b f_2\,\mathrm{d}\alpha$
(c) *$f \in \mathscr{R}(\alpha)$ on $[a,b]$, $a < c < b$*
$\Rightarrow \int_a^b f\,\mathrm{d}\alpha = \int_a^c f\,\mathrm{d}\alpha + \int_c^b f\,\mathrm{d}\alpha$
(d) *$f \in \mathscr{R}(\alpha)$ on $[a,b]$, $|f(x)| \leq M$ on $[a,b]$*
$\Rightarrow \left|\int_a^b f\,\mathrm{d}\alpha\right| \leq M[\alpha(b) - \alpha(a)]$
(e) *$f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$*
$\Rightarrow f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and $\int_a^b f\,\mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f\,\mathrm{d}\alpha_1 + \int_a^b f\,\mathrm{d}\alpha_2$
(e2) *$f \in \mathscr{R}(\alpha)$ and $c > 0$ const*
$\Rightarrow f \in \mathscr{R}(c\alpha)$ and $\int_a^b f\,\mathrm{d}(c\alpha) = c\int_a^b f\,\mathrm{d}\alpha$

6.13. **Theorem.** *$f, g \in \mathscr{R}(\alpha)$ on $[a,b] \Rightarrow$*
(a) $fg \in \mathscr{R}(\alpha)$
(b) $|f| \in \mathscr{R}(\alpha)$ and $\left|\int_a^b f\,\mathrm{d}\alpha\right| \le \int_a^b |f|\,\mathrm{d}\alpha$

6.14. **Definition.** *unit step function*
The *unit step function* is $I(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$

6.15. **Theorem.** *f bounded on $[a,b]$, continuous at $s \in (a,b)$, $\alpha(x) = I(x-s)$*
$\Rightarrow \int_a^b f\,\mathrm{d}\alpha = f(s)$

6.16. **Theorem.** *$c_n \ge 0$ for $n \in \mathbb{Z}_+$, $\sum c_n$ converges*
(s_n) is a sequence of distinct points in (a,b)
$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$
f continuous on $[a,b]$
$\Rightarrow \int_a^b f\,\mathrm{d}\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$
i.e. if α is a pure step function, the integral reduces to a finite or infinite series

6.17. **Theorem.** *α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a,b]$*
$f : [a,b] \to \mathbb{R}$ bounded
$f \in \mathscr{R}(\alpha) \Leftrightarrow f\alpha' \in \mathscr{R}$ in which case $\int_a^b f\,\mathrm{d}\alpha = \int_a^b f\alpha'\,\mathrm{d}x$

6.18. *Remark.* The two preceeding theorems make it possible in many cases to study series and integrals simultaneously, rather than separately
E.g. the moment of inertia of a straight wire of unit length, about an axis through an endpoint at right angles to the wire, is $\int_0^1 x^2\,\mathrm{d}m(x)$ where $m(x)$ is the mass contained in interval $[0,x]$
Special case: continuous density $\rho(x) = m'(x)$, then $\int_0^1 x^2 \rho(x)\,\mathrm{d}x$
Special case 2: all masses m_i concentrated at points x_i, then $\sum_i x_i^2 m_i$

6.19. **Theorem.** *change of variable*
$\phi : [A,B] \to [a,b]$ strictly increasing, continuous
$\alpha : [a,b] \to \mathbb{R}$ monotonically increasing
$f : [a,b] \to \mathbb{R}$, $f \in \mathscr{R}(\alpha)$
$\beta, g : [A,B] \to \mathbb{R}$, $\beta(y) = \alpha(\phi(y))$, $g(y) = f(\phi(y))$
$\Rightarrow g \in \mathscr{R}(\alpha)$ and $\int_a^b f\,\mathrm{d}\alpha = \int_A^B g\,\mathrm{d}\beta$

INTEGRATION AND DIFFERENTIATION.

Remark. Will consider real valued functions.
Integration and differentiation are, in some sense, inverse operations.

6.20. **Theorem.** *$f \in \mathscr{R}$ on $[a,b]$*
$F(x) = \int_a^x f(t)\,\mathrm{d}t$ for $a \le x \le b$
$\Rightarrow F$ continuous on $[a,b]$
Furthermore, f continuous at $x_0 \in [a,b]$
$\Rightarrow f$ differentiable at x_0 and $F'(x_0) = f(x_0)$

6.21. **Theorem.** *Fundamental theorem of calculus*
$f \in \mathscr{R}$ on $[a,b]$
$\exists F$ differentiable on $[a,b]$ s.t. $F' = f$
$\Rightarrow \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$

6.22. **Theorem.** *Integration by parts*
F, G differentiable on $[a,b]$
$F' = f \in \mathscr{R}$, $G' = g \in \mathscr{R}$
$\Rightarrow \int_a^b F(x)G(x)\,\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b f(g)G(x)\,\mathrm{d}x$

INTEGRATION OF VECTOR VALUED FUNCTIONS.

6.23. **Definition.** *Riemann integrable*
Let $f : [a,b] \to \mathbb{R}^k$ with components $f_1, \cdots, f_k$, $\alpha : [a,b] \to \mathbb{R}$ monotonically increasing
$f \in \mathscr{R}(\alpha)$ means each $f_j \in \mathscr{R}(\alpha)$ and $\int_a^b f\,\mathrm{d}\alpha = \left(\int_a^b f_1\,\mathrm{d}\alpha, \cdots, \int_a^b f_k\,\mathrm{d}\alpha\right)$

Remark. Theorems 6.12 a,c,e, 6.17, 6.20, and 6.21 remain valid

6.24. **Theorem.** *analogue of thm 6.21*
$f, F : [a,b] \to \mathbb{R}^k$, $f \in \mathscr{R}$ on $[a,b]$, $F' = f$
$\Rightarrow \int_a^b f(t)\,\mathrm{d}t = F(b) - f(a)$

6.25. **Theorem.** *extend thm 6.13b with norms, Schwarz*
$f : [a,b] \to \mathbb{R}^k$, $f \in \mathscr{R}(\alpha)$ for $\alpha : [a,b] \to \mathbb{R}$ monotonically increasing
$\Rightarrow |f| \in \mathscr{R}(\alpha)$ and $\left|\int_a^b f\,\mathrm{d}\alpha\right| \le \int_a^b |f|\,\mathrm{d}\alpha$

RECTIFIABLE CURVES.

Remark. This is a topic of geometric interest.
Useful for analytic functions of complex variable.

6.26. **Definition.** *curve, arc, closed curve, length of curve's polygonal path, length of curve, rectifiable*

- A *curve* $\gamma : [a,b] \to \mathbb{R}^k$ is a continuous mapping of an interval $[a,b]$
  note: curve is a mapping, not a point set, since different curves can have same range
- An *arc* is a 1-1 curve
- A *closed curve* γ is s.t. $\gamma(a) = \gamma(b)$
- The *length of a curve's polygonal path* wrt partition P of $[a,b]$ is $\Lambda(P,\gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})|$ where $\gamma(x_j)$ is a vertex
- The *length of curve* γ is $\Lambda(\gamma) = \sup_P \Lambda(P,\gamma)$
- γ is *rectifiable* means $\Lambda(\gamma) < \infty$

6.27. **Theorem.** *γ' continuous on $[a,b]$*
$\Rightarrow \gamma$ is rectifiable and $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t$

7. SEQUENCES AND SERIES OF FUNCTIONS

Remark. Will consider real and complex valued functions, but many theorems extend to vector valued functions, and even mappings into general metric spaces.

DISCUSSION OF THE MAIN PROBLEM.

7.1. **Definition.** *sequence of functions, pointwise convergence, limit, sum*

- A *sequence of functions* is $(f_n(x))$ where each f_n is defined on set E
- $(f_n(x))$ *converges pointwise* to *limit* $f(x)$ means $\lim_{n\to\infty} f_n(x) = f(x)$ for each x
- series $\sum f_n(x)$ converges pointwise to *sum of series* $f(x)$ means $\sum f_n(x) = f(x)$ for each x

Remark.

- what properties are preserved in the limit? Continuity, derivative, integral?
- Let f continuous at limit point x ie $\lim_{t\to x} f(t) = f(x)$
  Is the limit of a sequence of continuous functions also continuous?
  I.e. is $\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$?
- Following examples will show that limit process cannot in general be interchanged. Then will prove conditions under which limit process can be interchanged.

7.2. **Example.** *double sequence*
Let $s_{m,n} = \frac{m}{m+n}$, $m, n = 1, 2, \cdots$
$\forall n$ fixed, $\lim_{m\to\infty} s_{m,n} = 1$ i.e. $\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = 1$
$\forall m$ fixed, $\lim_{n\to\infty} s_{m,n} = 0$ i.e. $\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = 0$

7.3. **Example.** Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$, $n = 1, 2, \cdots$, $f(x) = \sum_{n=0}^{\infty} f_n(x)$
For $x = 0$, $f_n(0) = 0$ so $f(0) = 0$
For $x \ne 0$, $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = 1 + x^2$ since geometric series
So convergent sequence of continuous functions can have discontinuous sum

7.4. **Example.** Let $f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n}$ for $m = 1, 2, \cdots$
For $m!x \in \mathbb{Z}$, $f_m(x) = 1$
For $m!x \notin \mathbb{Z}$, $f_m(x) = 0$
Let $f(x) = \lim_{m\to\infty} f_m(x)$
For $x \in \mathbb{I}$, $f(x) = 0$ so $f(x) = 0$
For $x \in \mathbb{Q}, x = \frac{p}{q}$, $p, q \in \mathbb{Z}$, and $m \ge q \Rightarrow m!x \in \mathbb{Z}$ so $f(x) = 1$
So $\lim_{m\to\infty} \lim_{n\to\infty} (\cos m!\pi x)^{2n} = \begin{cases} 0 & x \in \mathbb{I} \\ 1 & x \in \mathbb{Q} \end{cases}$
So limit function is everywhere discontinuous

7.5. **Example.** Let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$, $x \in \mathbb{R}$, $n = 1, 2, \cdots$, $f(x) = \lim_{n\to\infty} f_n(x) = 0$
$f'(x) = 0$
$f_n'(x) = \sqrt{n}\cos nx$
At $x = 0$, $f'(0) = 0$ and $f_n'(0) \to +\infty$
So derivatives don't converge to derivative

7.6. **Example.** Let $f_n(x) = n^2 x(1-x^2)^n$, $x \in [0,1]$, $n = 1, 2, \cdots$
$\lim_{n\to\infty} f_n(x) = 0$ by theorem 3.20d on $(0,1]$
$n^2 \int_0^1 x(1-x^2)^n\,\mathrm{d}x = \frac{n^2}{2n+2} \to \infty$
$\int f = 0$
So limit of integral need not equal integral of limit

Remark. need stronger convergence

UNIFORM CONVERGENCE.

7.7. **Definition.** *uniform convergence*

- (f_n) *converges uniformly* on E to $f(x)$ means
  $\forall \epsilon > 0\ \exists N \in \mathbb{Z}$ s.t. $n \ge N \Rightarrow |f_n(x) - f(x)| \le \epsilon\ \forall x \in E$
  note: N must work $\forall x$
- $\sum_n f_n(x)$ *converges uniformly* on E means sequence of partial sums $(s_n(x))$, $s_n(x) = \sum_{i=1}^n f_i(x)$ converges uniformly on E

7.8. **Theorem.** *Cauchy criterion*
$(f_n(x))$ converges uniformly on $E \Leftrightarrow$
$\forall \epsilon \exists N$ s.t. $m, n \ge N, x \in E \Rightarrow |f_n(x) - f_m(x)| \le \epsilon$

7.9. **Theorem.** *follows from def 7.7*
$\lim_{n\to\infty} f_n(x) = f(x)$ on E, $M_n = \sup_{x\in E} |f_n(x) - f(x)|$
$f_n \to f$ uniformly on E
$\Leftrightarrow M_n \overset{n\to\infty}{\to} 0$

7.10. **Theorem.** *Weierstrass M-test*
$(f_n(x))$ on E, $|f_n(x)| \le M_n\ \forall x \in E$, $n = 1, 2, \cdots$
$\sum M_n$ converges
$\Rightarrow \sum f_n$ converges uniformly

UNIFORM CONVERGENCE AND CONTINUITY.

7.11. **Theorem.** *$f_n \to f$ uniformly on E in a metric space*
$\lim_{t\to x} f_n(t) = A_n$, $n = 1, 2, \cdots$, x is a limit point of E
$\Rightarrow (A_n)$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$
i.e. $\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$

7.12. **Theorem.** *Cor to thm 7.11*
(f_n) on E, each f_n continuous
$f_n \to f$ uniformly on E
$\Rightarrow f$ is continuous on E

Remark. When is converse true?

7.13. **Theorem.** *Dini's Theorem*
K compact
(f_n), each f_n continuous on K
$f_n \to f$ pointwise, f continuous on K
$f_n(x) \ge f_{n+1}(x)\ \forall x \in E$, $n = 1, 2, \cdots$
$\Rightarrow f_n \to f$ uniformly on K

7.14. **Definition.** *$\mathscr{C}(X)$, supremum norm*

- $\mathscr{C}(X)$ denotes the set of all complex valued, continuous, bounded functions with domain metric space X
  note: boundedness is redundent if X compact (thm 4.15)
- The *supremum norm* of each $f \in \mathscr{C}(X)$ is $\|f\| = \sup_{x \in X} |f(x)|$
  note: $f \in \mathscr{C}(X) \Rightarrow \|f\| < \infty$

Remark.

- supremum norm satisfies metric space axioms (2.15) between $f, g \in \mathscr{C}(X)$ since $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$ $\forall x$ so $\|f + g\| \leq \|f\| + \|g\|$
  So $\mathscr{C}(X)$ is a metric space
- closed subsets of $\mathscr{C}(X)$ are sometimes called *uniformly closed*
  The closure of a set $\mathscr{A} \subset \mathscr{C}(X)$ is called its *uniform closure*, etc.

7.15. **Theorem.** *The supremum norm metric makes $\mathscr{C}(X)$ into a complete metric space*

UNIFORM CONVERGENCE AND INTEGRATION.

7.16. **Theorem.** *α monotonically increasing on $[a, b]$*
$f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ for $n = 1, 2, \cdots$
$f_n \to f$ uniformly on $[a, b]$
$\Rightarrow f \in \mathscr{R}(\alpha)$ on $[a, b]$ and $\int_a^b f \, \mathrm{d}\alpha = \lim_{n\to\infty} \int_a^b f_n \, \mathrm{d}\alpha$

Corollary. *$f_n \in \mathscr{R}(\alpha)$ on $[a, b]$*
$f(x) = \sum_{n=1}^{\infty} f_n(x)$ uniformly on $[a, b]$
$\Rightarrow \int_a^b f \, \mathrm{d}\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, \mathrm{d}\alpha$
i.e. can integrate series term-by-term

UNIFORM CONVERGENCE AND DIFFERENTIATION.

7.17. **Theorem.**
(f_n), each f_n differentiable on $[a, b]$
$(f_n(x_0))$ converges for some $x_0 \in [a, b]$
$(f_n'(x))$ converges uniformly on $[a, b]$
$\Rightarrow f_n \to f$ uniformly on $[a, b]$ and $f'(x) = \lim_{n\to\infty} f_n'(x)$
note: proof much easier if also asume each f_n' continuous

7.18. **Theorem.** *$\exists\ f : \mathbb{R} \to \mathbb{R}$ continuous but nowhere differentiable*

EQUICONTINUOUS FAMILIES OF FUNCTIONS.

Remark. Want to generalize thm 3.6: every bounded sequence of complex numbers has a convergent subsequence.

7.19. **Definition.** *pointwise bounded, uniformly bounded*
(f_n) is *pointwise bounded* on E means $(f_n(x))$ is bounded $\forall x \in E$ i.e. $\exists$ finite valued function ϕ on E s.t. $|f_n(x)| < \phi(x)$, $x \in E$, $n = 1, 2, \cdots$
(f_n) is *uniformly bounded* on f means $\exists M$ s.t. $|f_n(x)| < M$ $\forall x \in E$, $n = 1, 2, \cdots$

Remark. Will see

- (f_n) pointwise bounded on E and $E_1 \subset E$ is countable
  can find subsequence (f_{n_k}) which converges $\forall x \in E$ (thm 7.23)
- (f_n) uniformly bounded, each f_n continuous on compact E
  there need not exist a sequence which converges pointwise on E

7.20. **Example.** Let $f_n(x) = \sin nx$ $x \in [0, 2\pi]$, $n = 1, 2, \cdots$
Suppose (will find contradiction) $\exists (n_k)$ s.t. $(\sin n_k x)$ converges $\forall x \in [0, 2\pi]$
Then $\lim_{k\to\infty} (\sin n_k x - \sin_{n_{k-1}})^2 = 0$
Then by Lebesgue's theorem (11.32), $\lim_{k\to\infty} \int_0^{2\pi} (\sin n_k x - \sin_{n_{k-1}})^2 \, \mathrm{d}x = 0$
But $\int_0^{2\pi} (\sin n_k x - \sin_{n_{k-1}})^2 = 2\pi$
Remark.

- every convergent sequence need not contain a unifromly convergent subsequence even if the sequence is uniformly bounded on compact set
- 7.6 shows sequence of bounded functions can converge without being uniformly bounded
- it is trivial to show uniform convergence of a sequence of bounded functions $\Rightarrow$ uniform boundedness

7.21. **Example.** Let $f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}$, $x \in [0, 1]$, $n = 1, 2, \cdots$
$|f_n(x)| \leq 1$ so (f_n) unformly bounded on $[0, 1]$
$\lim_{n\to\infty} f_n(x) = 0$ on $[0, 1]$ but $f_n(\frac{1)n}{=}1$, $n = 1, 2, \cdots$
So no subsequence can converge uniformly on $[0, 1]$

Remark. need concept of equicontinuity so that sequences of continuous functions converge uniformly

7.22. **Definition.** *equicontinuous*
Let family $\mathscr{F}$ of complex functions f on R in metric space X
$\mathscr{F}$ is *equicontinuous* on E means
$\forall \epsilon > 0$ $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \epsilon$ whenever $d_x(x, y) < \delta$, $x, y \in E$, $f \in \mathscr{F}$
note: every $f \in \mathscr{F}$ is uniformly continuous
note: ex 7.21 not equicontinuous

7.23. **Theorem.** *selection process*
(f_n) pointwise bounded sequence of complex funcitons on countable set E
$\Rightarrow f_n$ has a subsequence $(f_{n_k})(x)$ which converges $\forall x \in E$

7.24. **Theorem.** *K compact metric space, $f_n \in \mathscr{C}(K)$ for $n = 1, 2, \cdots$, (f_n) converges uniformly on K*
$\Rightarrow (f_n)$ equicontinuous on K

7.25. **Theorem.** *K compact, $f_n \in \mathscr{C}(K)$ for $n = 1, 2, \cdots$, (f_n) pointwise bounded and equicontinuous on K*
$\Rightarrow$ (a) (f_n) uniformly bounded on K
(b) (f_n) contains a uniformly convergent subsequence

STONE WEIERSTRASS THEOREM.

7.26. **Theorem.** *Weierstrass version*
f continuous conplex function on $[a, b]$
$\Rightarrow \exists$ sequence of polynomials P_n s.t. $P_n(0) = 0$ and $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a, a]$

7.27. **Corollary.** *for every interval $[-a, a]$ $\exists$ sequence of real polynomials P_n s.t. $P_n(0) = 0$ and $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a, a]$*

Remark. isolate properties of polynomials which make Weierstrass thm possible

7.28. **Definition.** *algebra, uniformly closed, uniform closure*

- An *algebra* is a family $\mathscr{A}$ of complex functions on set E s.t. i) $f + g \in \mathscr{A}$, ii) $fg \in \mathscr{A}$, and iii) $cf \in \mathscr{A}$ $\forall f, g \in \mathscr{A}$, $c \in \mathbb{C}$
  i.e. $\mathscr{A}$ is closed under addition, multiplication, and scalar multiplication
  note: for real f, $c \in \mathbb{R}$
- $\mathscr{A}$ is *uniformly closed* means $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A}$ and $f_n \to f$ uniformly on E
- The *uniform closure* of $\mathscr{A}$ is the set $\mathscr{B}$ of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$ (see def 7.14)

7.29. **Theorem.** *$\mathscr{B}$ is the uniform closure of algebra $\mathscr{A}$ of bounded functions*
$\Rightarrow \mathscr{B}$ is a uniformly closed algebra

7.30. **Definition.** *separate points, vanish at no point*

- A family of functions $\mathscr{A}$ on E on E means $\forall x_1, x_2 \in E$, $x_1 \neq x_2$, there corresponds a function $f \in \mathscr{A}$ s.t. $f(x_1) \neq f(x_2)$
- *$\mathscr{A}$ vanishes at no point* of E means to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ s.t. $g(x) \neq 0$

eg separate points and vanish at no point - algebra of all polynoials in one variable on $\mathbb{R}$
eg doesn't separate points - set of all even polynomials, say on $[-1, 1]$ since $f(-x) = f(x)$ for every even function

7.31. **Theorem.** *$\mathscr{A}$ algebra of functions on E separates points on and vanishes at not point of E*
$x_1, x_2 \in E$, $x_1 \neq x_2$, $x_1, c_2 \in \mathbb{C}$ (real for real agebra)
$\Rightarrow \mathscr{A}$ contains a function that $f(x_1) = c_1$ and $f(x_2) = c_2$

7.32. **Theorem.** *$\mathscr{A}$ algebra of real continuous functions on a compact set K*
$\mathscr{A}$ separates points on and vanishes at no point of K
$\Rightarrow$ uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K

7.33. **Theorem.** *Generalize to $\mathbb{C}$*
$\mathscr{A}$ is a self adjoint ($f \in \mathscr{A} \Rightarrow$ complex conjugate $\bar{f} \in \mathscr{A}$) algebra of complex continuous functions on a compact set K
$\mathscr{A}$ separates points on and vanishes at no point of K
$\Rightarrow$ uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K
i.e. $\mathscr{A}$ is dense in $\mathscr{C}(K)$