Parallel Computation of Quasigeostrophic Flow Over a Sphere Using Spectral Methods on Coupled Layers

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THESIS

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SUMMARY

The goal of this thesis is to model and efficiently compute the evolution of atmospheric synoptic-scale (hundreds to thousands of kilometers) cyclones and oceanic mesoscale (tens to hundreds of kilometers) eddies. Quasigeostrophic flow is a well suited model for these phenomena. This report consists of three chapters, (i) general equations, (ii) a derivation of the quasigeostrophic model, and (iii) computation of the quasigeostrophic model.

The first two chapters follow reference (1), henceforth referred to as "Pedlosky", although other reasonable references are (2) and (3), and more generally, (4) and (5). Because a general theory is desired before specializing, the first chapter concerns general models for fluid dynamics. These models couple variables, such as velocity, density, pressure, and temperature, through systems of partial differential equations. First a general model will be derived through physical laws such as conservation of mass, momentum, and energy. This model will be generalized to the perspective of a rotating basis and transformed to spherical coordinates. Because the general equations are difficult to compute, chapter two will derive an approximation called the quasigeostrophic model, which supports desired cyclone and eddy behavior. This derivation entails a formal scale analysis. The third chapter is devoted to computation of the quasigeostrophic model. The vertical coordinate will be discretized into layers. Each layer will be computed separately using spectral methods. Then each layer will be coupled with its neighboring layers. This process will be stepped forward in time. Although similar models have been computed before (6), the author's contribution is computing each layer in parallel, offering a computational speed-up.

This model may be of interest to anyone modeling fluid dynamics on a planet or even a star. Although the author's primary motivation is Earth's cyclones and eddies.

CHAPTER 1

GENERAL EQUATIONS

Where to start? Physically, fluid is discretized into particles. But it is said, (1), (2), (3), (4), (5), that particle interactions are approximately negligible when modeling large-scale dynamics. In this report, only large scale dynamics will be considered, so particle (discrete distributions of mass) dynamics can be approximated by continuum (continuous distribution of mass) dynamics; this is sometimes referred to as the continuum hypothesis. The continuum is over the independent variables position $\mathbf{x} = (x, y, z)^T$ and time t. The dependent variables of interest are

$$\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} u(\mathbf{x}, t) \\ v(\mathbf{x}, t) \\ w(\mathbf{x}, t) \end{pmatrix} = \text{velocity field}$$

$$p(\mathbf{x}, t) = \text{pressure field}$$

$$\rho(\mathbf{x}, t) = \text{density field}$$

$$T(\mathbf{x}, t) = \text{temperature field.}$$

Other variables will be defined as needed. To avoid technicalities, all values are assumed to be appropriate for the context.

Given initial and boundary conditions, how do these dependent variables evolve in time? In section 1.1, governing equations will be derived using conservation of mass, momentum, and energy, and a closure if necessary. The equations will be generalized to rotating basis in section 1.2 and spherical coordinates in section 1.3. Section 1.4 will derive the vorticity equations from the momentum equations.

1.1 General Fluid Dynamics Equations

[This section follows Pedlosky section 1.4.]

1.1.1 Conservation of Mass

Mass is neither created nor destroyed, only transported. So the local change in density is due to density flux,

$$\frac{\partial \rho}{\partial t} + \underbrace{\nabla \cdot (\rho \mathbf{u})}_{\text{density flux}} = 0$$

This can be rewritten

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \mathbf{u} = 0$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

is called the total (or material or Lagrangian) derivative. Intuitively, the total derivative is the time derivative of a parcel traveling with the velocity field along trajectory $x(t), y(t), z(t)$. The total derivative of any scalar or vector field $f(x(t), y(t), z(t), t)$ is, using chain rule,

$$\begin{aligned}\frac{\mathrm{d}f(x(t),y(t),z(t),t)}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial f}{\partial z} \\ &= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \\ &= \frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f,\end{aligned}$$

where the dot product is computed first for vector f and the gradient is computed first for scalar f. The total derivative allows all future equations to be interpreted with respect to a moving parcel, which deforms and changes volume, but with walls impervious to mass. This parcel is arbitrarily chosen, so the equation applies to all parcels simultaneously.

1.1.2 Conservation of Momentum

Newton's second law states that mass times acceleration of a parcel equals the sum of forces acting upon it. In the case of fluids, pressure gradient, body, and frictional forces act on a parcel, so the equation, per volume for convenience, is

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \underbrace{-\nabla p}_{\substack{\text{pressure gradient}\\ \text{force}}} + \underbrace{\rho\mathbf{g}}_{\text{body force}} + \underbrace{\mathscr{F}(\mathbf{u})}_{\text{frictional force}} .$$

The body force in this report will be gravity $\mathbf{g}$ towards the center of mass. The friction force will end up being negligible in this report, but will be retained symbolically for generality of the theory.

If density is constant, then the system is closed; four equations and four unknowns. Otherwise, an energy equation is needed.

1.1.3 Conservation of Energy and the Equation of State

The first law of thermodynamics states that energy is neither created nor destroyed, only transported and transformed between its various forms: kinetic, potential, thermal, etc. So change in internal energy of the parcel can be written

$$\rho\frac{\mathrm{d}e}{\mathrm{d}t} = \underbrace{-p\rho\frac{\mathrm{d}\rho^{-1}}{\mathrm{d}t}}_{\substack{\text{compressive}\\ \text{energy}}} + \underbrace{k\Delta T}_{\substack{\text{thermal diffusion}\\ \text{of energy}}} + \underbrace{\chi}_{\substack{\text{frictional dissipation}\\ \text{to thermal energy}}} + \underbrace{\rho Q}_{\substack{\text{internal heat}\\ \text{source energy}}}$$

where

$e =$ internal energy per unit mass

$k =$ thermal conductivity coefficient

$\chi =$ addition of heat due to viscous dissipation, negligible for us

$Q =$ rate of heat addition by internal heat sources, per unit mass.

This equation is too general for our purposes, and can be simplified. Pedlosky uses entropy, thermodynamic relations, and the equations of state $p = \rho RT$ for atmosphere and $\rho = \rho_0[1 - \alpha(T - T_0)]$ for ocean to obtain simplified energy equations

$$\begin{aligned} &\text{atmosphere :} && \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\theta}{C_p T}\left(\frac{k}{\rho}\Delta T + Q\right) \\ &\text{ocean :} && \frac{\mathrm{d}\rho}{\mathrm{d}t} = \kappa\Delta\rho + \frac{\alpha\rho_0}{C_p}Q \end{aligned}$$

where

$$\begin{aligned} \theta &= \text{potential temperature} \\ &= T\left(\frac{p_0}{p}\right)^{R/C_p} \\ p_0 &= \text{constant sea-level pressure} \\ R &= \text{universal gas constant} \\ C_p &= \text{specific heat at constant pressure} \\ Q &= \text{rate of internal heating} \\ k &= \text{thermal conductivity coefficient} \\ \kappa &= \text{thermal diffusivity coefficient} \\ &= \frac{k}{\rho C_p} \\ \alpha &= \text{thermal expansion coefficient.} \end{aligned}$$

The system is now complete with six equations, mass, momentum, energy, and state, in six unknowns, $\mathbf{u}, p, \rho, T$,

$$\left\{\begin{array}{ll} \text{mass} & \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\mathbf{u} \\ \text{momentum} & \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \frac{1}{\rho}\mathscr{F} \\ \text{energy} & \text{atmosphere}: \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\theta}{C_pT}\left(\frac{k}{\rho}\Delta T + Q\right) \\ & \text{ocean}: \frac{\mathrm{d}\rho}{\mathrm{d}t} = \kappa\Delta\rho + \frac{\alpha\rho_0}{C_p}Q \\ \text{state} & \text{atmosphere}: p = \rho RT \\ & \text{ocean}: \rho = \rho_0[1-\alpha(T-T_0)] \end{array}\right\}.$$

1.2 Rotating Equations

[This section follows Pedlosky sections 1.5 and 1.6.] Consider a planet rotating with constant angular velocity $\mathbf{\Omega} = \Omega\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is a constant unit vector. It is desired to rewrite the dynamics equations relative to a basis rotating with the planet.

Geometrically, a point $\mathbf{r}$ is independent of which basis it is described in. All scalar fields are unchanged, since their input is the same geometric point. All of the spacial derivatives used are invariant to rotation, except for non-Newtonian friction, which is outside of our scope. The time derivative of a scalar field is also unchanged (with some nuances which are irrelevant to this report, see Pedlosky). The only part that needs updating is the time derivative of a vector field, for which there exists a geometric relationship between a fixed ("inertial") basis I and a rotating basis R,

$$\underbrace{\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_I}_{\substack{\text{in inertial}\\ \text{basis}}} = \underbrace{\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_R}_{\substack{\text{in rotating}\\ \text{basis}}} + \underbrace{\boldsymbol{\Omega}\times}_{\substack{\text{imparted by}\\ \text{rotating basis}}} .$$

So velocity and acceleration are rewritten in the local basis,

$$\begin{aligned}
\mathbf{u}_I &= \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_I \\
&= \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_R + \boldsymbol{\Omega}\times\right]\mathbf{r} \\
&= \mathbf{u}_R + \boldsymbol{\Omega}\times\mathbf{r} \\
\frac{\mathrm{d}\mathbf{u}_I}{\mathrm{d}t} &= \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \\
&= \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_R + \boldsymbol{\Omega}\times\right]\left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_R + \boldsymbol{\Omega}\times\right]\mathbf{r} \\
&= \left[\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_R + 2\boldsymbol{\Omega}\times\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_R + \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times + \left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_R\times\right]\mathbf{r} \\
&= \underbrace{\left.\frac{\mathrm{d}\mathbf{u}_R}{\mathrm{d}t}\right|_R}_{\substack{\text{local}\\ \text{acceleration}}} + \underbrace{2\boldsymbol{\Omega}\times\mathbf{u}_R}_{\substack{\text{Coriolis}\\ \text{acceleration}}} + \underbrace{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})}_{\substack{\text{centripetal}\\ \text{acceleration}}} + \underbrace{\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\times\mathbf{r}}_{\substack{\text{rotating basis}\\ \text{acceleration}}} .
\end{aligned}$$

For our case, the rotating basis acceleration is zero since we assumed $\boldsymbol{\Omega}$ to be constant. Also, the centripetal acceleration can be rewritten as the gradient of a potential function and subsumed into the conservative body force term. But since centripetal acceleration is small (maximum $\Omega^2 r_0$ at the equator, where r_0 is the radius of the Earth) compared to gravity, the centripetal force will be neglected. Now the equations can be rewritten with respect to the rotating basis, with subscripts dropped for neatness since there is no ambiguity,

$$\left\{\begin{array}{ll} \text{mass} & \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\mathbf{u} \\ \text{momentum} & \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + 2\mathbf{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \frac{1}{\rho}\mathscr{F} \\ \text{energy} & \text{atmosphere} : \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\theta}{C_p T}\left(\frac{k}{\rho}\Delta T + Q\right) \\ & \text{ocean} : \frac{\mathrm{d}\rho}{\mathrm{d}t} = \kappa\Delta\rho + \frac{\alpha\rho_0}{C_p}Q \\ \text{state} & \text{atmosphere} : p = \rho RT \\ & \text{ocean} : \rho = \rho_0[1-\alpha(T-T_0)] \end{array}\right\}.$$

Notice that the only new term is the Coriolis term $2\mathbf{\Omega}\times\mathbf{u}$. When Ω is zero, the original equations are retained.

1.3 Spherical Coordinates

Consider the spherical domain with coordinates longitude $\theta \in [0, 2\pi]$, latitude $\phi \in [-\pi, \pi]$, and radius $r \in [0, \infty)$. Apply the change of variables

$$x = r\cos\theta\cos\phi$$

$$y = r\sin\theta\cos\phi$$

$$z = r\sin\phi$$

to the rotating equations to obtain the spherical rotating equations, copying Pedlosky equation 6.2.1-7,

$$
\left\{
\begin{array}{ll}
\text{mass} & \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{\partial w}{\partial r} + \frac{2w}{r} + \frac{1}{r\cos\theta}\frac{\partial(v\cos\theta)}{\partial\theta} + \frac{1}{r\cos\theta}\frac{\partial u}{\partial\phi}\right) = 0 \\
\text{momentum} & \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{uw}{r} - \frac{uv\tan\theta}{r} - 2\Omega\sin\theta v + 2\Omega\cos\theta w = -\frac{1}{\rho r\cos\theta}\frac{\partial p}{\partial\phi} + \frac{\mathscr{F}_\phi}{\rho} \\
 & \frac{\mathrm{d}v}{\mathrm{d}t} + \frac{wv}{r} + \frac{u^2\tan\theta}{r} + 2\Omega\sin\theta u = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \frac{\mathscr{F}_\theta}{\rho} \\
 & \frac{\mathrm{d}w}{\mathrm{d}t} - \frac{u^2+v^2}{r} - 2\Omega\cos\theta u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \frac{\mathscr{F}_r}{\rho} \\
\text{energy} & \text{atmosphere : } \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\theta}{C_p T}\left(\frac{k}{\rho}\Delta T + Q\right) \\
 & \text{ocean : } \frac{\mathrm{d}\rho}{\mathrm{d}t} = \kappa\Delta\rho + \frac{\alpha\rho_0}{C_p}Q \\
\text{closure} & \text{atmosphere : } p = \rho R T \\
 & \text{ocean : } \rho = \rho_0[1 - \alpha(T - T_0)]
\end{array}
\right\}
$$

where the total derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial}{\partial\phi} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial r}.$$

Like in any curvilinear coordinate system (e.g. cylindrical, spherical, oblate spheroidal), the new terms, called curvature terms, are due to the curvature of the coordinate system, and are negligible if velocities in the numerator are much less than the the denominator r. Also, coefficients for horizontal derivatives are called scale factors, and are also due to the the curvature of the coordinate system. The terms with $2\Omega\sin\theta$ and $2\Omega\cos\theta$ are the coriolis term in spherical coordinates.

1.4 Vorticity Equation

[Following Pedlosky section 2.4.] Define the vorticity ζ of a velocity field as its curl

$$\zeta = \nabla \times \mathbf{u}.$$

It would be useful to have an equation for the time-evolution of vorticity. Take the momentum equation in vector form,

$$\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla(gr) + \mathscr{F}$$

where r is the outward coordinate. Employ the vector identity $(\mathbf{u} \cdot \nabla)\mathbf{u} = \zeta \times \mathbf{u} + \frac{1}{2}\nabla|\mathbf{u}|^2$ and combine like terms,

$$\frac{\partial u}{\partial t} + (2\mathbf{\Omega} + \zeta) \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla(gr + \frac{1}{2}\,|\mathbf{u}|^2) + \mathscr{F}.$$

Take the curl of both sides,

$$\frac{\partial \zeta}{\partial t} + \nabla \times [(2\mathbf{\Omega} + \zeta) \times \mathbf{u}] = \frac{\nabla \rho \times \nabla p}{\rho^2} + \nabla \times \mathscr{F}.$$

Use vector identity $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$ and note that $\nabla \cdot (2\boldsymbol{\Omega} + \zeta) = 0$

$$\frac{\mathrm{d}(2\boldsymbol{\Omega} + \zeta)}{\mathrm{d}t} = \underbrace{((2\boldsymbol{\Omega} + \zeta) \cdot \nabla)\mathbf{u} - (2\boldsymbol{\Omega} + \zeta)\nabla \cdot \mathbf{u}}_{\text{vortex tilting and stretching}} + \underbrace{\frac{\nabla \rho \times \nabla p}{\rho^2}}_{\substack{\text{baroclinic production}\\ \text{of vorticity}}} + \underbrace{\nabla \times \mathscr{F}}_{\substack{\text{diffusion of}\\ \text{vorticity}}} .$$

The absolute vorticity $2\boldsymbol{\Omega} + \zeta$ is the sum of planetary vorticity $2\boldsymbol{\Omega}$ and relative vorticity ζ. When the domain is not spinning, $\boldsymbol{\Omega} = 0$, then these reduce to the standard vorticity equation. Vorticity can be transformed back, with Boit-Savart's law, to the non-divergent part of the velocity field. If the flow is divergence free, which is a reasonable approximation for the atmosphere and ocean, then the vorticity equation resolves the original velocity. The above equations will be used to derive the quasigeostrophic equation.

CHAPTER 2

DERIVATION OF QUASIGEOSTROPHIC EQUATIONS FOR A STRATIFIED FLUID ON A SPHERE

[This entire chapter follows Pedlosky sections 6.1-5, 6.8, and, where noted, Pedlosky section 2.8] The general equations stated above are resistant to analysis and computation. The motivation of this chapter is to obtain an approximation to the general model which supports desired cyclone and eddy phenomena. This approximation will be derived through a formal scale analysis – magnitudes of terms will be compared, and small terms will be neglected. After transforming this approximation to vorticity form and with simplifications from the energy equation, the quasigeostrophic equation will emerge. It will turn out that this equation is only valid on a tangent plane at midlatitudes. Along the way, other famous models will be visited including geostrophic wind, Brunt-Väisälä oscillations, and thermal wind.

2.1 General Equations

Take the spherical mass, momentum, energy, and state equations from section 1.3. In ansatz, it is convenient to change variables

$$x = \phi r_0 \cos\theta_0$$

$$y = (\theta - \theta_0) r_0$$

$$z = r - r_0$$

where θ_0 is a fixed midlatitude and r_0 is the surface radius. Derivatives become

$$\begin{aligned}
\frac{\partial}{\partial \phi} &= r_0 \cos\theta_0 \frac{\partial}{\partial x} \\
\frac{\partial}{\partial \theta} &= r_0 \frac{\partial}{\partial y} \\
\frac{\partial}{\partial r} &= \frac{\partial}{\partial z}
\end{aligned}$$

which allow future use of $\frac{r_0 \cos\theta_0}{r\cos\theta} = O(1)$ near midlatitudes. The rotating spherical equations become

$$\left\{\begin{array}{ll}
\text{mass} & \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{\partial w}{\partial r} + \frac{2w}{r} + \frac{r_0}{r\cos\theta}\frac{\partial(v\cos\theta)}{\partial y} + \frac{r_0\cos\theta_0}{r\cos\theta}\frac{\partial u}{\partial x}\right) = 0 \\
\text{momentum} & \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{uw}{r} - \frac{uv\tan\theta}{r} - 2\Omega\sin\theta v + 2\Omega\cos\theta w = -\frac{r_0\cos\theta_0}{\rho r\cos\theta}\frac{\partial p}{\partial x} + \frac{\mathscr{F}_x}{\rho} \\
 & \frac{\mathrm{d}v}{\mathrm{d}t} + \frac{wv}{r} + \frac{u^2\tan\theta}{r} + 2\Omega\sin\theta u = -\frac{r_0}{\rho r}\frac{\partial p}{\partial y} + \frac{\mathscr{F}_y}{\rho} \\
 & \frac{\mathrm{d}w}{\mathrm{d}t} - \frac{u^2+v^2}{r} - 2\Omega\cos\theta u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \frac{\mathscr{F}_z}{\rho} \\
\text{energy} & \text{atmosphere} : \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\theta}{C_p T}\left(\frac{k}{\rho}\Delta T + Q\right) \\
 & \text{ocean} : \frac{\mathrm{d}\rho}{\mathrm{d}t} = \kappa\Delta\rho + \frac{\alpha\rho_0}{C_p}Q \\
\text{closure} & \text{atmosphere} : p = \rho R T \\
 & \text{ocean} : \rho = \rho_0[1 - \alpha(T - T_0)]
\end{array}\right\}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{r_0 \cos\theta_0 u}{r\cos\theta}\frac{\partial}{\partial x} + \frac{r_0 v}{r}\frac{\partial}{\partial y} + w\frac{\partial}{\partial r}$$

$$\theta = \text{potential temperature}$$

$$= T\left(\frac{p_0}{p}\right)^{R/C_p}$$

$$p_0 = \text{constant sea-level pressure}$$

$$\rho_0 = \text{constant sea-level density}$$

$$R = \text{universal gas constant}$$

$$C_p = \text{specific heat at constant pressure}$$

$$Q = \text{rate of internal heating}$$

$$k = \text{thermal conductivity coefficient}$$

$$\kappa = \text{thermal diffusivity coefficient}$$

$$= \frac{k}{\rho C_p}$$

$$\alpha = \text{thermal expansion coefficient.}$$

2.2 Transformations Between Dimensional and Nondimensional Variables

The equations can be simplified by comparing magnitudes of terms and neglecting small terms. The following three sections are devoted to transforming the above equations into nondimensional form, where each term has an explicit magnitude. To transform between dimensional and nondimensional forms, all dimensional variables must be written as a product of constant scaling coefficients and nondimensional variables of $O(1)$ magnitude and variation.

Synoptic cyclones are characterized by length $L = 1000\text{km}$, velocity $U = 10\frac{\text{m}}{\text{s}}$, and depth $D = 10\text{km}$. Similarly for mesoscale eddies, $L = 100\text{km}$, $U = 0.05\frac{\text{m}}{\text{s}}$, and $D = 4\text{km}$. Pressure and density are subtle, and will be addressed in the next section. For neatness, henceforth denote all dimensional variables with a star $*$ (e.g. x_*) and nondimensional variables plainly (e.g. x). Now some transformations between dimensional and nondimensional variables can be defined,

$$
\begin{aligned}
x_* &= Lx \\
y_* &= Ly \\
z_* &= Dz \\
u_* &= Uu \\
v_* &= Uv \\
w_* &= \frac{UD}{L}w \\
\frac{\partial u_*}{\partial z_*} &= \frac{U}{D}\frac{\partial u}{\partial z} \\
t_* &= \frac{L}{U}t
\end{aligned}
$$

where vertical velocity scale $\frac{UD}{L}$ is found geometrically from a particle moving at speed U with a slope defined by distance D divided by L. In our case, the time scale is advective, $\frac{L}{U}$, since anything faster would be waves traveling faster than the characteristic parcel velocity. Some constants can be subsumed into important parameters,

$$
\begin{aligned}
\delta &= \text{aspect ratio} \\
&= \frac{D}{L} \\
\epsilon &= \text{Rossby number} \\
&= \frac{U}{fL} \\
\epsilon_T &= \text{temporal Rossby number} \\
&= \frac{U}{fL} \\
F &= \frac{f^2 L^2}{gD} \\
f &= \text{Coriolis parameter} \\
&= 2\Omega \sin\theta \quad \text{(which is } O(0) \text{ near the equator)} \\
f_0 &= \text{Coriolis parameter at latitude } \theta_0 \\
&= 2\Omega \sin\theta_0.
\end{aligned}
$$

The Rossby number and temporal Rossby number are equal for advective time scales $\frac{L}{U}$, but need not be equal in general. The friction terms will be retained only symbolically, since Pedlosky chapters 4 and 5 discuss how friction is negligible except for the boundary ("Eckman") layer. This thesis will consider only flow away from the boundary. The interested reader can see Pedlosky chapter 6.6.7,9 for generalization of the quasigeostrophic model to include the Eckman Layer.

2.3 Geostrophic Scaling of Pressure and Density

[Some parts of this section are also from Pedlosky section 2.8, where the Rossby numbers ϵ, ϵ_T are missing a factor of $\sin\theta$, which is $O(0.5)$ for midlatitudes so this doesn't grossly affect the approximations. After this section, the Rossby number will resume its above defined form.]

To avoid introducing new constants, it is desired to write magnitudes of pressure and density in terms of the existing constants. It is convenient to first rewrite the density and pressure as the sum of steady and perturbed parts, in dimensional form,

$$p_*(x_*, y_*, z_*, t_*) = p_s(z_*) + \tilde{p}_*(x_*, y_*, z_*, t_*)$$

$$\rho_*(x_*, y_*, z_*, t_*) = \rho_s(z_*) + \tilde{\rho}_*(x_*, y_*, z_*, t_*)$$

where the steady parts $p_s(z_*), \rho_s(z_*)$ are known functions that satisfy the z-momentum equation with $\mathbf{u}_* = 0$, in dimensional and nondimensional forms,

$$\frac{\partial p_s}{\partial z_*} = -\rho_s g$$

$$\frac{1}{D}\frac{\partial p_s}{\partial z} = -\rho_s g.$$

Now plug all available nondimensional variables into the dimensional momentum equations to obtain a mixed dimensional-nondimensional equation

$$\frac{U}{T}\frac{\partial \mathbf{u}}{\partial t} + \frac{U^2}{L}(\mathbf{u}\cdot\nabla)\mathbf{u} + 2\Omega\hat{\mathbf{k}}\times U\mathbf{u} = -\frac{1}{\rho_*}\nabla p_* + g\hat{\mathbf{k}} + \mathscr{F}.$$

Pressure and density are the only nondimensional parts, and will be solved for. Friction is still considered negligible, but is retained symbolically for generality of the equations. Dividing each term by $2\Omega U$, the coefficients of some terms can be replaced by important parameters,

$$\epsilon_T\frac{\partial \mathbf{u}}{\partial t} + \epsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \hat{\mathbf{k}}\times\mathbf{u} = -\frac{1}{2\Omega U\rho_*}\nabla p_* + \frac{g}{2\Omega U}\hat{\mathbf{k}} + Ek\mathscr{F}.$$

For all flows with small ϵ_T, ϵ, δ, and Ek (where Ek is the Eckman parameter, see Pedlosky for more complete exposition), which includes the desired cyclones and eddies, the coriolis and pressure gradient terms balance with error $O[\max(\epsilon_T, \epsilon, \delta, Ek)]$),

$$\begin{aligned}
-\underbrace{2\Omega\sin\theta}_{f}Uv + \cancelto{O(\delta)}{2\Omega\cos\theta\frac{UD}{L}w} &= -\frac{r_0\cos\theta_0}{\rho_* r_*\cos\theta L}\frac{\partial\tilde{p}_*}{\partial x} \\
\underbrace{2\Omega\sin\theta}_{f}Uu &= -\frac{r_0}{\rho_* r_* L}\frac{\partial\tilde{p}_*}{\partial y} \\
\cancelto{O(\delta)}{-2\Omega\cos\theta Uu} &= -\frac{1}{\rho_*}\frac{\partial p_*}{\partial z_*} - g.
\end{aligned}$$

These are called the geostrophic equations. The second coriolis term in the x-momentum equation is negligible since its magnitude $O(2\Omega U\delta)$ is much less, at midlatitudes, than the first coriolis term of magnitude $O(2\Omega U)$. To recover the magnitude of pressure, approximate $\frac{r_0}{r_*} = O(1)$ and for midlatitudes $\frac{\cos\theta_0}{\cos\theta} = O(1)$. Isolate $\tilde{p}_*$ in the horizontal momentum equations to obtain its magnitude

$$\tilde{p}_* = O(\rho_* f_0 U L).$$

In the z-momentum equation, the Coriolis term of $O(\rho 2\Omega U)$ can be neglected since it is negligible compared to the pressure gradient of $O(\rho 2\Omega U L/D)$. Cancel the static parts from the buoyancy and pressure gradient, since they must balance, then isolate $\tilde{\rho}_*$ and find its magnitude

$$\begin{aligned}\tilde{\rho}_* &= \frac{1}{gD}\frac{\partial \tilde{p}_*}{\partial z} \\ &= O\left(\frac{\tilde{p}_*}{gD}\right) \\ &= O\left(\frac{\rho_* f_0 U L}{gD}\right) \\ &= O(\rho_* \epsilon F).\end{aligned}$$

Pedlosky argues that $\tilde{\rho}_*/\rho_* \leq O(\epsilon)$ for our chosen L and D. So $\tilde{\rho}_* \ll \rho_s$. So the magnitudes and nondimensional transformations perturbed pressure are

$$\tilde{p}_* = O(\rho_s U f_0 L)$$

$$\tilde{p}_* = \rho_s U f_0 L \tilde{p}$$

and similarly for density

$$\tilde{\rho}_* = O\left(\rho_s \epsilon F\right)$$

$$\tilde{\rho}_* = \rho_s \epsilon F \tilde{\rho}$$

where newly defined nondimensional variables $\tilde{p}, \tilde{\rho} = O(1)$ with $O(1)$ variation.

So the transformations of density and pressure to nondimensional form are

$$p_* = p_s + \rho_s U f_0 L \tilde{p}$$

$$\rho_* = \rho_s (1 + \epsilon F \tilde{\rho}).$$

It should be stressed that our assumption $\frac{\cos\theta_0}{\cos\theta} = O(1)$ is not valid near the poles. Also, the geostrophic velocity is not valid at the equator since solving for geostrophic velocity requires dividing by f which is zero at the equator. So our equations are valid only at midlatitudes.

2.4 Governing Equations in Nondimensional Form

Nondimensional equations are obtained by plugging nondimensional transformations of all variables. Note that $\frac{r_*}{r_0} = 1 + \delta \frac{L}{r_0} z$ since $z_* = r_* - r_0$. Plugging,

$$\left\{ \begin{array}{ll} \text{mass} & \epsilon F \frac{\mathrm{d}\rho}{\mathrm{d}t} + (1 + \epsilon F \rho) \left(\frac{w}{\rho_s} \frac{\partial \rho_s}{\partial z} + \frac{\partial w}{\partial z} + 2 \frac{D}{r_*} w + \frac{\partial v}{\partial y} \frac{r_0}{r_*} - \frac{L}{r_*} v \tan\theta + \frac{r_0}{r_*} \frac{\cos\theta_0}{\cos\theta} \frac{\partial u}{\partial x} \right) = 0 \\ \text{momentum} & \epsilon \left[\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{L}{r_*} \left(\delta u w - u v \tan\theta \right) \right] - v \frac{\sin\theta}{\sin\theta_0} + \delta w \frac{\cos\theta}{\sin\theta_0} = - \frac{\cos\theta_0}{\cos\theta} \frac{r_0}{r_*} \frac{\partial p}{\partial x} \frac{1}{1 + \epsilon F \rho} + \frac{F_{*x}}{\rho_* f_0 U} \\ & \epsilon \left[\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{L}{r_*} \left(\delta v w + u^2 \tan\theta \right) \right] + u \frac{\sin\theta}{\sin\theta_0} = - \frac{r_0}{r_*} \frac{\partial p}{\partial y} \frac{1}{1 + \epsilon F \rho} + \frac{F_{*y}}{\rho_* f_0 U} \\ & (1 + \epsilon F \rho) \left[\epsilon \delta^2 \frac{\mathrm{d}w}{\mathrm{d}t} - \frac{\epsilon \delta L}{r_*} (u^2 + v^2) - \frac{\delta u \cos\theta}{\sin\theta_0} \right] = - \frac{1}{\rho_s} \frac{\partial (p \rho_s)}{\partial z} - \rho + \frac{F_{*\phi}}{\rho_s f_0 U} \delta \end{array} \right\}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\cos\theta_0}{\cos\theta}\frac{r_0}{r_*}\frac{\partial}{\partial x} + v\frac{r_0}{r_*}\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

Nondimensional energy equations will be derived later. These are the full nondimensional equations. The order of each term is measured by its coefficients which include parameters $\epsilon, \delta, \frac{L}{r_0}, F$. Because some terms have magnitude that is a combination of these parameters, their relative magnitudes are important when comparing terms. For example, $\frac{L\epsilon}{r_0}$ is important.

2.5 Magnitudes and Relative Magnitudes of Nondimensional Parameters

Needed are magnitudes for $\epsilon, \delta, \frac{L}{r_0}, F$, and their relative magnitudes. But first, in anticipation of the tangent ("β") plane domain, define the northward gradient of f at θ_0,

$$\begin{aligned} \beta_0 &= \frac{1}{r_0}\left.\frac{\partial f}{\partial \theta}\right|_{\theta_0} \\ &= \frac{2\Omega\cos\theta_0}{r_0} \end{aligned}$$

and define the useful nondimensional parameter $\beta = \frac{\beta_0 L^2}{U}$.

At midlatitudes, $f_0 = 10^{-4}\frac{1}{\mathrm{s}}$ and $\beta_0 = 10^{-13}\frac{1}{\mathrm{cm}\cdot\mathrm{s}}$. For atmospheric synoptic cyclones, motion is characterized by $L = 10^3$km, $U = 10\frac{\mathrm{m}}{\mathrm{s}}$, and $D = 10$km, so nondimensional parameters are

$$\epsilon = 10^{-1}, \quad F = 10^{-1}, \quad \delta = 10^{-1}, \quad \frac{L}{r_0} = 10^{-1}, \quad \beta = 1.$$

For oceanic mesoscale eddies, motion is characterized by $L = 100\text{km}$, $U = 5\frac{\text{cm}}{\text{s}}$, $D = 4\text{km}$ so the nondimensional parameters are

$$\epsilon = 5 \cdot 10^{-3}, \quad F = 0.2 \cdot 10^{-2}, \quad \delta = 2 \cdot 10^{-2}, \quad \frac{L}{r_0} = 10^{-2}, \quad \beta = 0.5.$$

So to approximate these phenomena, choose magnitudes

$$\epsilon = O(\frac{L}{r_0}) = O(F) = O(\delta) \ll 1$$

and choose relative magnitudes

$$\frac{\delta}{\epsilon}, \frac{\frac{L}{r_0}}{\epsilon}, \frac{F}{\epsilon} = O(1)$$

even as $\epsilon \to 0$. Other useful relative magnitudes are

$$\begin{aligned} \frac{r_*}{r_0} - 1 &= O(\frac{\delta L}{r_0}) \\ &= O(\epsilon^2) \end{aligned}$$

and

$$\begin{aligned} \frac{\beta_0 L}{\epsilon f_0} &= \frac{L}{\epsilon r_0} \cot \theta_0 \\ &= O(\frac{L}{\epsilon r_0}). \end{aligned}$$

2.6 Asymptotic Expansions of Independent Variables and of Trigonometric Functions

Every dependent variable can be expanded in nondimensional parameters $\epsilon, \delta, \frac{L}{r_0}, F$, e.g. $u(x, y, z, t, \epsilon, \delta, \frac{L}{r_0}, F)$. But since these parameters are of the same order of magnitude, it is sufficient to expand only in ϵ,

$$u(x, y, z, t, \epsilon) = u_0(x, y, z, t) + \epsilon u_1(x, y, z, t) + \epsilon^2 u_2(x, y, z, t) + O(\epsilon^3)$$

$$v(x, y, z, t, \epsilon) = v_0(x, y, z, t) + \epsilon v_1(x, y, z, t) + \epsilon^2 v_2(x, y, z, t) + O(\epsilon^3)$$

$$w(x, y, z, t, \epsilon) = w_0(x, y, z, t) + \epsilon w_1(x, y, z, t) + \epsilon^2 w_2(x, y, z, t) + O(\epsilon^3)$$

$$p(x, y, z, t, \epsilon) = p_0(x, y, z, t) + \epsilon p_1(x, y, z, t) + \epsilon^2 p_2(x, y, z, t) + O(\epsilon^3)$$

$$\rho(x, y, z, t, \epsilon) = \rho_0(x, y, z, t) + \epsilon \rho_1(x, y, z, t) + \epsilon^2 \rho_2(x, y, z, t) + O(\epsilon^3)$$

where each $u_k, v_k, w_k, p_k, \rho_k$ is $O(1)$ with $O(1)$ variation.

Also, in anticipation of sphericity accounted for by a tangent plane at midlatitudes, expand trigonometric functions in asymptotic series of about midlatitude θ_0,

$$\sin\theta = \sin\theta_0 + \frac{L}{r_0} y \cos\theta_0 - \left(\frac{L}{r_0}\right)^2 \frac{y^2}{2} \sin\theta_0 + O\left[\left(\frac{L}{r_0}\right)^3\right]$$
$$\cos\theta = \cos\theta_0 - \frac{L}{r_0} y \sin\theta_0 + \left(\frac{L}{r_0}\right)^2 \frac{y^2}{2} \cos\theta_0 + O\left[\left(\frac{L}{r_0}\right)^3\right]$$
$$\tan\theta = \tan\theta_0 + \frac{L}{r_0} y \frac{1}{\cos^2\theta_0} + \left(\frac{L}{r_0}\right)^2 y^2 \frac{\tan\theta_0}{\cos^2\theta_0} + O\left[\left(\frac{L}{r_0}\right)^3\right].$$

2.7 The $O(1)$ Momentum Equations

Plug the asymptotic expansions of dependent variables and trigonometric functions into the nondimensional equations. The $O(1)$ equations, i.e. with error $O(\epsilon)$, are

$$\left\{\begin{array}{ll} \text{mass} & \frac{1}{\rho_s}\frac{\partial(w_0\rho_s)}{\partial z} + \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y} = 0 \\ \text{momentum} & v_0 = \frac{\partial p_0}{\partial x} \\ & u_0 = -\frac{\partial p_0}{\partial y} \\ & \rho_0 = -\frac{1}{\rho_s}\frac{\partial(\rho_s p_0)}{\partial z} \end{array}\right\}.$$

As expected from geostrophic scaling, these are the geostrophic equations. The equations appear to be in rectangular coordinates. Horizontal velocities in dimensional units are

$$\begin{aligned} v_{*0} &= \frac{1}{\rho_s f_0}\frac{\partial p_{*0}}{\partial x_*} \\ u_{*0} &= -\frac{1}{\rho_s f_0}\frac{\partial p_{*0}}{\partial y_*}. \end{aligned}$$

The only influence of sphericity is f_0.

Consider the mass equation. The horizontal divergence is zero, so

$$\frac{\partial(w_0\rho_s)}{\partial z} = 0.$$

So $w_0\rho_s(z)$ must be independent of z. Vertical velocity at the flat rigid boundary is zero, $w_0|_{z=0} = 0$, therefore it is everywhere zero, $w_0 = 0$. (For nonviscous flow, the vertical velocity due to Eckman pumping also turns out to be negligible. And along small surface topography, vertical velocity is also negligible. See Pedlosky for details.) So $w_0 = 0$ and therefore

$$w = \epsilon w_1(x,y,z,t) + \epsilon^2 w_2(x,y,z,t) + O(\epsilon^3).$$

2.8 The $O(\epsilon)$ Momentum Equations

Because the $O(1)$ equations are diagnostic (independent of time), it is desired to find a prognostic (time-dependent) equation for p_0 (and thence u_0, v_0, ρ_0 using the $O(1)$ momentum equations). Perhaps the $O(\epsilon)$ equations will reveal something. The $O(\epsilon)$ equations are

$$\left\{\begin{array}{ll} \text{mass} & \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} - \frac{L}{\epsilon r_0} v_0 \tan\theta_0 + \frac{L}{\epsilon r_0} y \tan\theta_0 \frac{\partial u_0}{\partial x} + \frac{1}{\rho_s}\frac{\partial(\rho_s w_1)}{\partial z} = 0 \\ \text{momentum} & \frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} - v_1 - v_0\left(\frac{L}{\epsilon r_0}\right) y\cot\theta_0 = -\frac{\partial p_1}{\partial x} - \frac{Ly}{\epsilon r_0}\tan\theta_0\frac{\partial p_0}{\partial x} \\ & \frac{\partial v_0}{\partial t} + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + u_1 + u_0\left(\frac{L}{\epsilon r_0}\right) y\cot\theta_0 = -\frac{\partial p_1}{\partial y}. \end{array}\right\}$$

The $\frac{L}{\epsilon r_0}$ terms on the left side of the x, y momentum equations are due to variations of the Coriolis parameter with latitude, and correspond to the tangent ("β") plane. The $\frac{L}{\epsilon r_0}$ term on the right side of the x momentum equation is from variations in the metric term $\cos\theta$ relating longitudinal variations and eastward length changes. This extra term makes the spherical equations different from purely β-plane equations. However, this term will be killed in the vorticity equation, so sphericity is accounted for entirely by the β-plane.

2.9 The $O(\epsilon)$ Vorticity Equation

The $O(\epsilon)$ vorticity equation is found by cross-differentiating the horizontal momentum equations and using $w_0 = 0$, $\frac{\beta_0 L}{f_0} = \frac{L}{r_0}\cot\theta_0$, and $\beta = \frac{\beta_0 L^2}{U}$

$$\frac{\partial \zeta_0}{\partial t} + u_0\frac{\partial \zeta_0}{\partial x} + v_0\frac{\partial \zeta_0}{\partial y} + \beta v_0 = \frac{L}{\epsilon r_0}\tan\theta_0\frac{\partial p_0}{\partial x} + \frac{L}{\epsilon r_0}y\tan\theta_0\frac{\partial^2 p_0}{\partial x \partial y} - \frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y}$$

where

$$\zeta_0 = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}.$$

Plug in the $O(1)$ mass equation and geostrophic velocities u_0, v_0, cancel terms to obtain

$$\frac{\mathrm{d}_0}{\mathrm{d}t}(\zeta_0 + \beta y) = \frac{1}{\rho_s}\frac{\partial(\rho_s w_1)}{\partial z}.$$

If w_1 can somehow be written in terms of p_0 (or in terms of variables recoverable from p_0, e.g. $u_0, v_0, \rho_0, \zeta_0$), then the $O(\epsilon)$ vorticity equation can be used to propagate p_0 in time. Determining w_1 will require understanding static stability as well as the thermodynamic equations, which are the topics of the next two sections.

It should be noted that sphericity is entirely accounted for by the β-plane, and the equations are in rectangular coordinates on the β-plane.

2.10 Vertically Displaced Parcel, Brunt-Väisälä frequency oscillations

[Following Pedlosky section 6.4, this section will use dimensional notation, for neatness. The remaining sections will return to nondimensional notation.]

The purpose of this section is to define the Brunt-Väisälä function $N(z)$ to be used later. Consider a fluid at rest, $\mathbf{u} = 0$. So the pressure surfaces p_s align with the density surfaces ρ_s. Displace a parcel at arbitrary height z by a small vertical distance h reversibly (slow enough so that pressure adjusts continuously) and adiabatically (fast enough so thermal dissipation or heat addition are negligible).

First consider the atmospheric case. The energy equation says that potential temperature is conserved. First obtain density in terms of potential temperature,

$$\begin{aligned}
\theta &= T\left(\frac{p|_{z=0}}{p}\right)^{\frac{R}{C_p}} \\
&= T\left(\frac{p|_{z=0}}{p}\right)^{\frac{C_p - C_v}{C_p}} && \text{since } R = C_p - C_v \\
&= \frac{T}{p} p|_{z=0} \left(\frac{p}{p|_{z=0}}\right)^{\frac{1}{\gamma}} && \text{where } \gamma = \frac{C_p}{C_v} \\
&= \frac{p|_{z=0}}{R\rho} \left(\frac{p}{p|_{z=0}}\right)^{\frac{1}{\gamma}} && \text{since } p = \rho R T \\
\rho &= \frac{p|_{z=0}}{R\theta} \left(\frac{p}{p|_{z=0}}\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

Employing Newton's second law, mass times acceleration of the parcel equals the buoyancy force, all per volume for convenience,

$$\begin{aligned}
\rho\frac{\partial^2 h}{\partial t^2} &= g\left[\underbrace{\left(\cancel{\rho_s|_{h=0}} + \frac{\partial \rho}{\partial z}h\right)}_{\substack{\text{density of}\\ \text{displaced parcel}}} - \underbrace{\left(\cancel{\rho_s|_{h=0}} + \frac{\partial \rho_s}{\partial z}h\right)}_{\substack{\text{ambient density}\\ \text{at height } h}}\right] \\
&= g\left[\frac{\partial}{\partial z}\left(\frac{p|_{z=0}}{R\theta}\left(\frac{p}{p|_{z=0}}\right)^{\frac{1}{\gamma}}\right)h - \frac{\partial \rho_s}{\partial z}h\right] \\
&\approx g\left[\frac{1}{\gamma}\frac{1}{p}\underbrace{\frac{p|_{z=0}}{R\theta}\left(\frac{p}{p|_{z=0}}\right)^{\frac{1}{\gamma}}}_{\rho}\frac{\partial p}{\partial z} - \frac{\partial \rho_s}{\partial z}\right]h \qquad \text{since } \theta \approx \text{const for } h \text{ small} \\
\frac{\partial^2 h}{\partial t^2} &\approx g\left[\frac{1}{\gamma}\frac{\partial \ln p}{\partial z} - \frac{\partial \ln \rho}{\partial z}\right]h \qquad \text{since } \rho \approx \rho_s \text{ for } h \text{ small} \\
&= g\frac{1}{\theta}\frac{\partial \theta}{\partial z}h \qquad \text{since } \frac{\partial \ln \theta}{\partial z} = \frac{1}{\gamma}\frac{\partial \ln p}{\partial z} - \frac{\partial \ln \rho}{\partial z} \\
&= N^2 h
\end{aligned}$$

where the the (Brunt-Väisälä) function $N(z) = \sqrt{\frac{g}{\theta}\frac{\partial \theta}{\partial z}}$ was defined. Intuitively, the buoyancy force on the parcel is proportional to its distance from equilibrium, like a mass-spring system. In the case that $\frac{\partial \theta}{\partial z} > 0$, the force is restoring and the solution of the ODE is oscillations with frequency $N(z)$, i.e. $h(z) = a\cos(Nt) + b\sin(Nt)$ where constants a, b depend on initial conditions.

For the ocean, compressibility is approximately negligible, so the buoyancy force is written in terms of density. Pedlosky's observes that for the ocean, $\frac{\partial \rho_s}{\partial z} \approx 0$, so the ODE becomes

$$\frac{\partial^2 h}{\partial t^2} = N^2 h$$

where $N(z) = \sqrt{-\frac{g}{\rho}\frac{\partial \rho}{\partial z}}$. In the case that dense fluid underlies light fluid, i.e. $\frac{\partial \rho}{\partial z} < 0$, the solution is oscillations at frequency $N(z)$.

Pedlosky displays graphs of observed values of $N(z)$ for both ocean and atmosphere, so it will be assumed that these functions are known. Finally, it is noted that the oscillation frequency $N(z)$ is much larger than earth rotation frequency 2Ω, so dynamics may have multiple time scales.

2.11 Atmosphere: Nondimensional Energy Equation and Recovering w_1

The goal is to write w_1 in terms of p_0 (or $O(1)$ variables derivable from p_0 such as u_0, v_0, ρ_0). Perhaps the energy equation will be useful. To obtain the nondimensional energy equation, we need a transformation of potential temperature θ_* [for the remainder of the chapter, asterisk $*$ means dimensional variable] to nondimensional form θ. Then ϵ-series expansions will provide the $O(1)$ part of potential temperature in terms of p_0. Finally, w_1 will be emerge in terms of p_0, as desired.

2.11.1 Nondimensional Transformation of Potential Temperature

Take the logarithm of the dimensional potential temperature $\theta_* = \frac{p_*|_{z=0}}{R\rho_*}\left(\frac{p_*}{p_*|_{z=0}}\right)^{1/\gamma}$

$$\ln\theta_* = \frac{1}{\gamma}\ln p_* - \ln\rho_* + \underbrace{\frac{R}{C_p}\ln p_*|_{z=0} - \ln R}_{\text{constant}}.$$

Plug in the nondimensional transformations of pressure $p_* = p_s + \rho_s U f_0 L p$ and density $\rho_* = \rho_s(1 - \epsilon F\rho)$, define the constant $c = \frac{R}{C_p}\ln p_*|_{z=0} - \ln R$,

$$\ln\theta_* = \frac{1}{\gamma}(p_s + \rho_s U f_0 L p) - \ln[\rho_s(1 + \epsilon F\rho)] + c.$$

Use $\ln(a+b) = \ln a + \ln(1 + \frac{b}{a})$ and properties of the logarithm,

$$\ln\theta_* = \frac{1}{\gamma}\ln p_s + \frac{1}{\gamma}\ln\left(1 + \frac{\rho_s U f_0 L p}{p_s}\right) - \ln\rho_s - \ln(1 + \epsilon F\rho) + c.$$

Use $\ln\theta_s(z) = \frac{1}{\gamma}\ln p_s - \ln\rho_s + c$, use the Taylor series $\ln a = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \ldots$ on $0 < a < 2$, and use $\epsilon F = \frac{U f_0 L}{gD}$,

$$\ln\theta_* = \ln\theta_s + \frac{1}{\gamma}\frac{\rho_s U f_0 L p}{p_s} - \epsilon F\rho + O((\epsilon F)^2).$$

Pedlosky is now guided to say that, just like density, potential temperature is the sum of its static value and some small deviation of $O(\epsilon F)$,

$$\theta_* = \theta_s(1 + \epsilon F\theta).$$

Also, to be used later, θ up to $O(\epsilon F)$ error is

$$\theta = \frac{\rho_s p G D}{\gamma p_s} - \rho.$$

2.11.2 Nondimensional Energy Equation

Finally, we can obtain the energy equation in nondimensional form. Plug transformation $\theta_* = \theta_s(1 + \epsilon F\theta)$ into the dimensional energy equation from section 2.1 to obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{w}{\epsilon F\theta_s}\frac{\partial \theta_s}{\partial z}(1 + \epsilon F\theta) = \frac{\theta_*}{\theta_s}\left(\frac{H_*}{C_p T_*}\right)\frac{gD}{U^2 f_0}$$

where the total heating rate of each fluid element is defined

$$H_* = \frac{k}{\rho_*}\Delta T_* + Q_*.$$

The transformation of H_* to nondimensional form is needed. The magnitude is found from tropospheric observations $U = O(10\text{m/s})$, $D = O(10\text{km})$, $C_pT_* = O(gD) = O(200\text{cm}^2/\text{s}^2)$ so $H_* = O(U^2 f_0) = O(10\text{cm}^2/\text{s}^3)$. So the nondimensional transform of H_* is

$$H = H_* \frac{gD}{C_p T_* f_0 U^2}.$$

2.11.3 Expanding in ϵ series and keeping $O(1)$ terms

Take the nondimensional energy equation. Plug ϵ series expansions of density, pressure, and potential temperature $\theta = \theta_0 + \epsilon\theta_1 + O(\epsilon^2)$, and keep $O(1)$ terms

$$\frac{\mathrm{d}_0\theta_0}{\mathrm{d}t} + w_1 S(z) = H$$

where the stratification parameter is defined

$$\begin{aligned} S(z) &= \frac{F^{-1}}{\theta_s}\frac{\partial\theta_s}{\partial z} \\ &= \frac{N_s^2}{f_0^2}\frac{D^2}{L^2}. \end{aligned}$$

The magnitude of $S(z)$ is $O(1)$ since $F = O(\epsilon)$ and $\frac{1}{\theta_s}\frac{\partial\theta_s}{\partial z} = O(\epsilon)$. Recall $N_s(z) = \frac{g}{\theta_s}\frac{\mathrm{d}\theta_s}{\mathrm{d}z_*}$ is the Brunt-Väisälä frequency of the rest-state. Finally the thermodynamic equation is solved for w_1 to obtain

$$w_1 = \frac{1}{S(z)}\left(-\frac{\mathrm{d}_0\theta_0}{\mathrm{d}t} + H\right).$$

For our model to work, θ_0 and H must be a function of p_0 or any $O(1)$ fields recoverable from p_0. The next subsection will derive θ_0 in terms of p_0. And H can depend on θ_0 and the $O(1)$ motion field, but Pedlosky argues it is negligible since heat accession over advective time scale $\frac{L}{U}$ is small.

2.11.4 Finding θ_0 in terms of p_0

To write w_1 is in terms of p_0, we need θ_0 in terms of p_0. Recall $\theta = -\rho + \frac{1}{\gamma}\left(\frac{\rho_s g D}{p_s}\right) p$. Plug the ϵ series expansions of density, pressure, and potential temperature. Solve for θ_0,

$$\theta_0 = -\rho_0 + \frac{1}{\gamma}\left(\frac{\rho_s g D}{p_s}\right) p_0.$$

Plug in the $O(1)$ mass equation $\rho_0 = -\frac{1}{\rho_s}\frac{\partial(\rho_s p_0)}{\partial z}$ and use the nondimensional hydrostatic relation $\rho_s g = \frac{1}{D}\frac{\partial p_s}{\partial z}$,

$$\begin{aligned}\theta_0 &= \frac{1}{\rho_s}\frac{\partial(\rho_s p_0)}{\partial z} + \frac{1}{\gamma}\frac{1}{p_s}\frac{\partial p_s}{\partial z} p_0 \\ &= \frac{\partial p_0}{\partial z} - p_0\left(\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} - \frac{1}{\gamma p_s}\frac{\partial p_s}{\partial z}\right) \\ &= \frac{\partial p_0}{\partial z} - p_0 \frac{1}{\theta_s}\frac{\partial \theta_s}{\partial z}.\end{aligned}$$

The second term on the right hand side is negligible, since in the atmosphere with $D = 10$km, $\frac{1}{\theta_s}\frac{\partial \theta_s}{\partial z} = \frac{D}{\theta_s}\frac{\partial \theta_s}{\partial z_*} = O(10^{-1}) = O(\epsilon)$. So to $O(\epsilon)$,

$$\theta_0 = \frac{\partial p_0}{\partial z}.$$

This can be considered a compact form of the hydrostatic relation.

2.12 Atmosphere: Quasigeostrophic Equation

This completes the goal, and the quasigeostrophic equation can be written solely in terms of only p_0 and known functions. Plug w_1 into the quasigeostrophic equation, and simplify the vortex stretching term,

$$\begin{aligned}\frac{1}{\rho_s}\frac{\partial(\rho_s w_1)}{\partial z} &= \frac{1}{\rho_s}\frac{\partial}{\partial z}\left[\frac{\rho_s}{S}\left(H - \frac{\mathrm{d}_0\theta_0}{\mathrm{d}t}\right)\right] \\ &= \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s H}{S}\right) - \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\mathrm{d}_0\theta_0}{\mathrm{d}t}\right) \\ &= \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s H}{S}\right) - \frac{1}{\rho_s}\frac{\mathrm{d}_0}{\mathrm{d}t}\left[\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\theta_0\right)\right] - \frac{1}{S}\left(\frac{\partial u_0}{\partial z}\frac{\partial \theta_0}{\partial x} + \frac{\partial v_0}{\partial z}\frac{\partial \theta_0}{\partial y}\right).\end{aligned}$$

The rightmost two terms cancel after plugging the thermal wind equations (which are a combination of geostrophic equations $v_0 = \frac{\partial p_0}{\partial x}$, $u_0 = -\frac{\partial p_0}{\partial y}$ and hydrostatic equation $\theta_0 = \frac{\partial p_0}{\partial z}$)

$$\begin{aligned}\frac{\partial u_0}{\partial z} &= -\frac{\partial \theta_0}{\partial y} \\ \frac{\partial v_0}{\partial z} &= \frac{\partial \theta_0}{\partial x}.\end{aligned}$$

Now the quasigeostrophic equation can be written neatly, bringing one of vertical stretching terms into the time derivative,

$$\frac{\mathrm{d}_0}{\mathrm{d}t}\left[\zeta_0 + \beta y + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\theta_0\right)\right] = \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s H}{S}\right).$$

An even neater version of the quasigeostrophic equation is after $u_0, v_0, \theta_0, \zeta_0, H$ have all been plugged in, and following tradition by using streamfunction notation $p_0 = \psi$, neglecting heating term H,

$$\left[\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial y}\right]\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \beta y + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial \psi}{\partial y}\right)\right] = 0.$$

Now p_0 can be propagated forward in time with the quasigeostrophic equation, and the relevant variables can be recovered through

$$\begin{aligned}
u_0 &= \frac{\partial p_0}{\partial x} \\
v_0 &= -\frac{\partial p_0}{\partial y} \\
\theta_0 &= \frac{\partial p_0}{\partial z} \\
\rho_0 &= -\frac{1}{\rho_s}\frac{\partial(\rho_s p_0)}{\partial z} \\
w_1 &= \frac{1}{S}\left(H(p_0) - \frac{\mathrm{d}_0\theta_0}{\mathrm{d}t}\right) \\
H &= H(p_0) \\
&\approx 0 \quad \text{for advective time scales.}
\end{aligned}$$

This quasigeostrophic equation is a generalization of the quasigeostrophic equations for the shallow water equations, with stratification accounted for by the vortex stretching terms. If stratification is strong, then $S \to \infty$, then $w_1 \to 0$, and the equations become similar to the barotropic form, $\frac{\mathrm{d}_0}{\mathrm{d}t}(\zeta_0 + \beta y) = 0$. If stratification is weak, $S \to 0$, then the resulting model is a homogeneous model discussed in Pedlosky chapter 5.

Finally, the equations can be converted back to dimensional units,

$$\left[\frac{\partial}{\partial t} + u_* \frac{\partial}{\partial x_*} + v_* \frac{\partial}{\partial y_*}\right] \left[\zeta_* + \beta_0 y_* + \frac{1}{\rho_s} \frac{\partial}{\partial z_*} \left(\frac{\rho_s f_0^2}{N_s^2} \frac{\partial \tilde{p}_*}{\partial z}\right)\right] = 0$$

where

$$\begin{aligned}
\tilde{p}_* &= p_*(x, y, z, t) - p_s(z) \\
u_* &= \frac{1}{\rho_s f_0} \frac{\partial p_*}{\partial x} \\
v_* &= -\frac{1}{\rho_s f_0} \frac{\partial p_*}{\partial y} \\
\zeta_* &= \frac{1}{\rho_s f_0} \left(\frac{\partial^2}{\partial x_*^2} + \frac{\partial^2}{\partial y_*^2}\right) \tilde{p}_*.
\end{aligned}$$

Notice that the dimensional geostrophic relations only involve constant Coriolis parameter f_0. The only effect of sphericity is the linear planetary vorticity $\beta_0 y_*$.

2.13 Ocean: Nondimensional Energy Equation and Recovering w_1

Recall the thermodynamic equation for the ocean,

$$\frac{\mathrm{d}\rho_*}{\mathrm{d}t_*} = \kappa \Delta \rho_* + \frac{\alpha \rho_0}{C_p} Q.$$

Transform to nondimensional by plugging in $\rho_* = \rho_s(1 + \epsilon F \rho)$,

$$\epsilon F \frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{w}{\rho_s}\frac{\partial \rho_s}{\partial z}(1 + \epsilon F \rho) = -\frac{H_* L}{U}$$

where the total heating rate of each fluid element is defined

$$H_* = -\frac{\kappa \Delta \rho_*}{\rho_s} + \frac{\alpha \rho_0}{\rho_s C_p} Q.$$

Plug ϵ-series expansions and keep $O(1)$ terms,

$$-\frac{\mathrm{d}_0 \rho_0}{\mathrm{d}t} + w_1 S = H$$

where

$$\begin{aligned} H &= \frac{gD}{U^2 f_0} H_* \\ S(z) &= \frac{N_s^2 D^2}{f_0^2 L^2} \\ &= O(1) \\ N_s &= \left(-\frac{g}{\rho_s}\frac{\partial \rho_s}{\partial z_*} \right)^{\frac{1}{2}}. \end{aligned}$$

Solve $O(1)$ energy equation for w_1

$$w_1 = \frac{H}{S} + \frac{1}{S}\frac{\mathrm{d}_0 \rho_0}{\mathrm{d}t}.$$

This is similar to the atmospheric case, with potential temperature replaced with density.

2.14 Ocean: Quasigeostrophic Equation

Plug w_1 into vortex stretching term, simplify

$$
\begin{aligned}
\frac{1}{\rho_s}\frac{\mathrm{d}(\rho_s w_1)}{\mathrm{d}z} &= \frac{1}{\rho_s}\left[\frac{\rho_s}{S}\left(H - \frac{\mathrm{d}_0\rho_0}{\mathrm{d}t}\right)\right] \\
&= \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s H}{S}\right) - \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\mathrm{d}_0\rho_0}{\mathrm{d}t}\right) \\
&= \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s H}{S}\right) - \frac{1}{\rho_s}\frac{\mathrm{d}_0}{\mathrm{d}t}\left[\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\rho_0\right)\right] + \frac{1}{S}\cancelto{0}{\left(\frac{\partial u_0}{\partial z}\frac{\partial \rho_0}{\partial x} + \frac{\partial v_0}{\partial z}\frac{\partial \rho_0}{\partial y}\right)}.
\end{aligned}
$$

The rightmost two terms cancel after plugging the thermal wind equations (which are a combination of geostrophic equations $v_0 = \frac{\partial p_0}{\partial x}$, $u_0 = -\frac{\partial p_0}{\partial y}$ and hydrostatic equation $\theta_0 = \frac{\partial p_0}{\partial z}$)

$$
\begin{aligned}
\frac{\partial u_0}{\partial z} &= -\frac{\partial \theta_0}{\partial y} \\
\frac{\partial v_0}{\partial z} &= \frac{\partial \theta_0}{\partial x}.
\end{aligned}
$$

Recall O(1) hydrostatic approximation, simplify

$$
\begin{aligned}
\rho_0 &= -\frac{1}{\rho_s}\frac{\partial \rho_s p_0}{\partial z} \\
&= -\frac{\partial p_0}{\partial z} - \cancelto{O(\epsilon)}{\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z}} \\
&= -\frac{\partial p_0}{\partial z}
\end{aligned}
$$

where Pedlosky uses observed function ρ_s to show $\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} = O(\epsilon)$. Plugging into the $O(\epsilon)$ vorticity equation, the quasigeostrophic equation is derived,

$$\frac{\mathrm{d}_0}{\mathrm{d}t}\left[\zeta_0 + \beta_y - \frac{\partial}{\partial z}\left(\frac{\rho_0}{S}\right)\right] = \frac{\partial}{\partial z}\left(\frac{H}{S}\right).$$

Plugging everything for the streamfunction $\psi = p_0$ and neglecting heating,

$$\left[\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial y}\right]\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \beta_y - \frac{\partial}{\partial z}\left(\frac{1}{S}\frac{\partial \psi}{\partial z}\right)\right] = 0.$$

Finally, the equations can be converted back to dimensional units.

$$\left[\frac{\partial}{\partial t} + u_{0*}\frac{\partial}{\partial x_*} + v_{0*}\frac{\partial}{\partial y_*}\right]\left[\zeta_{0*} + \beta_0 y_* + \frac{\partial}{\partial z_*}\left(\frac{f_0^2}{N_s^2}\frac{\partial \tilde{p}_{0*}}{\partial z}\right)\right] = 0$$

where

$$\begin{aligned}
\tilde{p}_* &= p_*(x, y, z, t) - p_s(z) \\
u_{0*} &= \frac{1}{\rho_s f_0}\frac{\partial p_{0*}}{\partial x} \\
v_{0*} &= -\frac{1}{\rho_s f_0}\frac{\partial p_{0*}}{\partial y} \\
\zeta_{0*} &= \frac{1}{\rho_s f_0}\left(\frac{\partial^2}{\partial x_*^2} + \frac{\partial^2}{\partial y_*^2}\right)\tilde{p}_{0*}.
\end{aligned}$$

Notice that the dimensional geostrophic relations only involve constant Coriolis parameter f_0. The only effect of sphericity is the linear planetary vorticity $\beta_0 y_*$.

CHAPTER 3

COMPUTATION OF THE QUASIGEOSTROPHIC MODEL ON A SPHERE

The quasigeostrophic equation derived above will now be computed. First the equations will be rewritten more generically to accommodate different versions of the quasigeostrophic equations (1), (6), (7), (8). Then the model will be discretized in the vertical, in the horizontal, and in time. Next, a parallelization scheme will be proposed. Finally, some numerical experiments will be performed to experimentally measure the speedup of the parallel scheme. The actual behavior of the simulated quasigeostrophic equations is outside the scope of this report.

3.1 Discretizing the Quasigeostrophic Equation

Recall the quasigeostrophic equation derived in chapter 2. Many versions of this equation exist, with different vertical coupling term, different forcing, and different vertical coordinate. Rewrite the quasigeostrophic equation generically, using z as the vertical coordinate,

$$\frac{\partial q}{\partial t} = - \underbrace{\mathbf{u}_\psi \cdot \nabla q}_{\substack{\text{advection of}\\ \text{vorticity}}} + \underbrace{F}_{\substack{\text{forcing}\\ \text{and damping}}}$$

where

$$
\begin{aligned}
q(\mathbf{x},t) &= \text{total vorticity, everywhere outward so treated as scalar field} \\
&= \underbrace{\Delta\psi}_{\text{local vorticity}} + \underbrace{f}_{\substack{\text{vorticity from} \\ \text{rotating planet}}} + \underbrace{g(z)\frac{\partial}{\partial z}\left(h(z)\frac{\partial\psi}{\partial z}\right)}_{\substack{\text{vortex stretching} \\ \text{(“vertical coupling”)}}} \\
\psi(\mathbf{x},t) &= \text{horizontal streamfunction at each level} \\
\mathbf{u}_\psi(\mathbf{x},t) &= \text{horizontal velocity due to the } O(1) \text{ pressure} \\
&= \nabla^\perp\psi \\
f &= \text{coriolis parameter} \\
&= 2\Omega\sin\theta \text{ for sphere} \\
g(z), h(z) &= \text{known functions} \\
F &= \text{known constants or a function of } \psi.
\end{aligned}
$$

Intuitively, the time change $\frac{\partial q}{\partial t}$ of a fluid parcel's total vorticity is a sum of advection, forcing, and damping. The velocity $\mathbf{u}_\psi$ and total vorticity q can be plugged into the equation to yield one equation in one unknown ψ,

$$
\frac{\partial}{\partial t}\left[\Delta\psi + f + g(z)\frac{\partial}{\partial z}\left(h(z)\frac{\partial\psi}{\partial z}\right)\right] = -\nabla^\perp\psi\cdot\nabla\left[\Delta\psi + f + g(z)\frac{\partial}{\partial z}\left(h(z)\frac{\partial\psi}{\partial z}\right)\right] + F
$$

but we will use variable q for neatness. Next a domain must be chosen. The equations were derived to support the midlatitude β-plane, but it can be advantageous to use a spherical domain to avoid horizontal boundary conditions. (Although putting this model on a sphere may be controversial, Marshall (6) claims the spherical model to be realistic at the poles and tropics and therefore relevant to the meteorological community.) In spherical coordinates, position is $\mathbf{x} = (\phi, \theta, z)$, where $\phi \in [0, 2\pi)$ is longitude, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is latitude, and $z \in [z_N, z_0]$ is vertical layer. The coriolis term for the sphere is $f = 2\Omega \sin\theta$. The horizontal derivative operators are

$$\begin{aligned}
\nabla &= \hat{\phi}\frac{1}{cos\theta}\frac{\partial}{\partial\phi} + \hat{\theta}\frac{\partial}{\partial\theta} \\
\nabla^{\perp} &= -\hat{\phi}\frac{\partial}{\partial\theta} + \hat{\theta}\frac{1}{cos\theta}\frac{\partial}{\partial\phi} \\
\Delta &= \frac{1}{\cos\theta}\left[\frac{1}{\cos\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right)\right].
\end{aligned}$$

A trick to simplify the advection term is changing latitudinal coordinate to $\mu = \sin\theta$, $\mu \in (-1, 1)$, $\frac{\partial}{\partial\mu} = \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}$ to yield

$$\begin{aligned}
\nabla^{\perp}\psi \cdot \nabla q &= \frac{1}{\cos\theta}\left(\frac{\partial\psi}{\partial\phi}\frac{\partial q}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial q}{\partial\phi}\right) \\
&= \frac{\partial\psi}{\partial\phi}\frac{\partial q}{\partial\mu} - \frac{\partial\psi}{\partial\mu}\frac{\partial q}{\partial\phi}
\end{aligned}$$

so the advection term looks exactly like it would in rectangular coordinates.

3.1.1 Time discretization

The system is explicit for q, so once the right hand side is computed, the model can be stepped forward in time. A reasonable choice for time-stepping is second order Runge-Kutta.

3.1.2 Vertical Discretization

Observing the equation, it seems simplest to first discretize the vertical coordinate, since the only term which needs vertical discretization is the vertical coupling term. This term can be approximated numerically by discretizing the vertical coordinate into $N+1$ layers, each of constant height z_i and of thickness Δz. Intuitively, flow is in horizontal layers which are coupled through this vertical coupling term. The quasigeostrophic equation becomes

$$\begin{aligned}\frac{\partial q_i}{\partial t} &= -\left(\frac{\partial \psi_i}{\partial \phi}\frac{\partial q_i}{\partial \mu} - \frac{\partial \psi_i}{\partial \mu}\frac{\partial q_i}{\partial \phi}\right) + F_i \\ q_i &= \Delta\psi_i + 2\Omega\sin\theta + \left[g(z)\frac{\partial}{\partial z}\left(h(z)\frac{\partial \psi}{\partial z}\right)\right]_i\end{aligned}$$

where the subscript $i \in \{0, 1, 2, \ldots, N\}$ denotes the vertical layer. To approximate vertical coupling, rewrite the outer vertical derivative using centered differencing with half-steps,

$$g(z)\frac{\partial}{\partial z}\left(h(z)\frac{\partial \psi}{\partial z}\right)\bigg|_{z=z_i} \approx \frac{g_i}{\Delta z}\left[\left(h(z)\frac{\partial \psi}{\partial z}\right)\bigg|_{z=z_{i+1/2}} - \left(h(z)\frac{\partial \psi}{\partial z}\right)\bigg|_{z=z_{i-1/2}}\right].$$

Approximate the inner vertical derivatives using a half-step scheme

$$\begin{aligned}\frac{\partial \psi}{\partial z}\bigg|_{z=z_{i+1/2}} &\approx \frac{1}{\Delta z}(\psi_{i+1} - \psi_i) \\ \frac{\partial \psi}{\partial z}\bigg|_{z=z_{i-1/2}} &\approx \frac{1}{\Delta z}(\psi_i - \psi_{i-1}).\end{aligned}$$

Combine,

$$g(z)\frac{\partial}{\partial z}\left(h(z)\frac{\partial \psi}{\partial z}\right)\bigg|_{z=z_i} \approx \frac{g_i}{(\Delta z)^2}\left[h_{i+1/2}\psi_{i+1} + h_{i-1/2}\psi_{i-1} - \left(h_{i+1/2} + h_{i-1/2}\right)\psi_i\right].$$

Specify the vertical coupling at the lower and upper boundaries

$$\frac{\partial}{\partial z}\left(h(z)\frac{\partial \psi}{\partial z}\right)\bigg|_{z=z_0} \approx \frac{g_0 h_{1/2}}{(\Delta z)^2}(\psi_1 - \psi_0)$$
$$\frac{\partial}{\partial z}\left(h(z)\frac{\partial \psi}{\partial z}\right)\bigg|_{z=z_N} \approx \frac{g_N h_{N-1/2}}{(\Delta z)^2}(\psi_{N-1} - \psi_N).$$

The weight of the discretization, $R_{i\pm 1/2} := \frac{\Delta z}{\sqrt{g_i h_{i\pm 1/2}}}$, is in some references known as the Rossby radius of deformation.

Intuitively, on each layer, the equations become barotropic vorticity equations which interact with each other through coupling with neighbors.

3.1.3 Horizontal Discretization

A decision must be made for how to discretize each horizontal (barotropic) part. Two difficult horizontal computations are computing the advection term and computing the laplacian. So what to do? Traditional finite difference schemes are known to produce numerical vorticity. Instead, consider the spectral space in which it is known that the laplacian operation is a trivial multiplication by eigenvalues. The spectrum of eigenvectors and eigenvalues is found by solving Laplace's equation $\Delta f = 0$, $f : \mathbb{R}^2 \to \mathbb{R}$ with separation of variables. For a periodic rectangular domain, the eigenfunctions are sines and cosines. For our spherical domain, the eigenfunctions are spherical harmonics

$$Y_m^n(\phi, \mu) = P_n^m(\mu)\frac{e^{im\phi}}{\sqrt{2\pi}}$$

over the spectrum $0 \leq |m| \leq n$ where the normalized Legendre functions are

$$P_n^m(\mu) = \frac{1}{2^n n!}\sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}(1-\mu^2)^m \frac{\partial^{m+n}}{\partial \mu^{m+n}}(1-\mu^2)^n.$$

The eigenvalues corresponding to eigenfunctions $Y_n^m(\phi, \mu)$ are

$$\Lambda_n^m = -n(n+1)$$

which depend only on n (since the dependence on m is in both parts of the eigenfunctions, a peculiarity of spherical harmonics). Spherical harmonics are orthonormal since combining orthonormality of Fourier and Legendre parts yields orthonormality of their products

$$\int_{-1}^{1} \mathrm{d}\mu P_{n'}m(\mu)P_{n''}^m(\mu) = \delta_{n',n''}$$

$$\int_0^{2\pi} \mathrm{d}\phi \frac{e^{im'\phi}}{\sqrt{2\pi}}\frac{e^{im''\phi}}{\sqrt{2\pi}} = \delta_{m',m''}$$

$$\int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\mu Y_{n'}^{m'}(\phi, \mu)Y_{n''}^{m''}(\phi, \mu) = \delta_{n',n'',m',m''}.$$

So the spherical harmonics form a complete orthonormal basis for the space of square integrable functions on the unit sphere $L^2(S^2)$. A function $f(\phi, \mu)$ defined on a sphere can be expanded in ("projected onto") spherical harmonic series,

$$f(\phi, \mu) = \sum_{n=0}^{\infty} \sum_{|m|<n} \hat{f}_n^m Y_n^m(\phi, \mu)$$

where the coefficients (which are also a function of time t if f is a function of time) are found using orthogonality

$$\hat{f}_n^m = \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\mu f(\phi, \mu)[Y_n^m(\mu, \phi)]^\dagger.$$

But in computation, computer memory can hold a finite number of spectral coefficients, so functions f will be projected onto a finite number of spherical harmonics truncated by some M,

$$f_\Lambda(\phi, \mu) = \sum_{m,n}^{M} \hat{f}_n^m Y_n^m(\phi, \mu).$$

This is an approximation of the function f, with exactness if f is defined on this finite set, or if the truncation is lifted. Now the desired result is achieved, the laplace and inverse laplace operators are

$$\Delta^k f_\Lambda(\phi, \mu) = \sum_{m,n}^{M} [-n(n+1)]^k \hat{f}_n^m Y_n^m(\mu, \phi)$$

$$\Delta^{-k} f_\Lambda(\phi, \mu) = \sum_{m,n}^{M} \frac{1}{[-n(n+1)]^k} \hat{f}_n^m Y_n^m(\mu, \phi)$$

with the inverse laplacian meaningful only if the spacial mean is zero, $\bar{f}(\mu, \phi) = 0$.

3.1.4 Transforming Between q and ψ

Now the transformation between total vorticity q and streamfunction ψ can be performed. In spectral space, for each fixed m, n,

$$\hat{\mathbf{q}}_m^n = \begin{pmatrix} & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \dots & 0 & R_{i-1/2}^{-2} & [-n(n+1) - R_{i-1/2}^{-2} - R_{i-1/2}^{-2}] & R_{i+1/2}^{-2} & 0 & \dots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \end{pmatrix} \hat{\psi}_m^n + \hat{\mathbf{f}}_m^n$$

where vectors $\hat{\mathbf{q}}_m^n, \hat{\psi}_m^n$ have a component for each layer, and $\hat{\mathbf{f}}_m^n$ is the transform of the coriolis term $2\Omega \sin\theta$ which is independent of vertical layer. The reverse transform from $\hat{\mathbf{q}}_m^n$ to $\hat{\psi}_m^n$ can be done by the standard tridiagonal reverse substitution method. This is the only computation with vertical coupling; all other computations can be done separately for each layer.

3.1.5 Advection Term

Consider the advection term $\nabla^\perp \psi \cdot \nabla q = \frac{\partial\psi}{\partial\phi}\frac{\partial q}{\partial\mu} - \frac{\partial\psi}{\partial\mu}\frac{\partial q}{\partial\phi}$. In spectral space, the ϕ-derivatives are a multiplication of each spectral coefficient by im. The μ-derivatives are complicated in spectral space, involving derivatives of Legendre functions which are not orthogonal. Instead, it is computationally advantageous to compute the advection term in physical space, then transform back to spectral space, since all other computations can be done in spectral space. The mapping to physical space at each (yet to be decided) physical space grid point (ϕ, μ) is

$$
\begin{aligned}
\frac{\partial}{\partial \phi} f_\Lambda(\phi, \mu) &= \sum_{n=0}^{M} \sum_{|m|<n} \hat{f}_n^m \frac{\partial}{\partial \phi} Y_n^m(\phi, \mu) \\
&= \sum_{n=0}^{M} \sum_{|m|<n} \hat{f}_n^m P_n^m(\mu) \frac{\partial}{\partial \phi} \frac{e^{im\phi}}{\sqrt{2\pi}} \\
&= \sum_{n=0}^{M} \sum_{|m|<n} im \hat{f}_n^m P_n^m(\mu) \frac{e^{im\phi}}{\sqrt{2\pi}} \\
&= \sum_{m=0}^{M} \frac{e^{im\phi}}{\sqrt{2\pi}} \left(im \sum_{n=m}^{M} \hat{f}_n^m P_n^m(\mu) \right) \\
\frac{\partial}{\partial \mu} f_\Lambda(\phi, \mu) &= \sum_{n=0}^{M} \sum_{|m|<n} \hat{f}_n^m \frac{\partial}{\partial \mu} Y_n^m(\phi, \mu) \\
&= \sum_{n=0}^{M} \sum_{|m|<n} \hat{f}_n^m \frac{e^{im\phi}}{\sqrt{2\pi}} \frac{\partial}{\partial \mu} P_n^m(\mu) \\
&= \sum_{m=0}^{M} \frac{e^{im\phi}}{\sqrt{2\pi}} \left(\sum_{n=m}^{M} \hat{f}_n^m \frac{\partial}{\partial \mu} P_n^m(\mu) \right)
\end{aligned}
$$

where the μ-derivatives of Legendre functions are precomputed at each grid point μ and stored in memory. The final rearrangement of each sum reveals that Discrete Fourier Transform (DFT) can be employed for the latitudinal part. Once the advection term is computed at each grid point in physical space, it is transformed back to spectral space using

$$
\begin{aligned}
\hat{f}_n^m &= \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\mu f(\phi, \mu) [Y_n^m(\mu, \phi)]^\dagger \\
&= \int_0^{2\pi} \frac{e^{im\phi}}{\sqrt{2\pi}} \left(\int_{-1}^{1} f(\phi, \mu) P_n^m(\mu) \mathrm{d}\mu \right) \mathrm{d}\phi
\end{aligned}
$$

where the rearrangement reveals that Discrete Fourier Transform (DFT) can be employed for the latitudinal part.

The integral over ϕ is exact using the Discrete Fourier Transform (DFT) and requires the latitudinal grid points to be equidistant. The integral in μ can also be done exactly using Gauss-Legendre quadrature. So the μ-integral is exactly the a weighted sum

$$\int_{-1}^{1} f(\mu)\mathrm{d}\mu = \sum_{j=1}^{M} f(\mu_j)w_j$$
$$w_j = \frac{2(1-\mu_j^2)}{M^2(\mu P_M(\mu_j) - P_{M-1}(\mu_j))^2}$$

where, for each truncation M used, grid points μ_j are roots of the Mth-order Legendre polynomial $P_M(\mu)$ (i.e. n=M, m=0), precomputed using Newton iteration. And weights w_j are precomputed from the given formula. Both the forward and backward transformations are exact, with the latitudinal parts done quickly using DFT. Once the advection term is computed and transformed back to spectral space, the forcing and damping terms can be added.

3.1.6 Forcing and Damping Term

A damping term can account for viscosity, which is significant only at the ground ("Eckman") layer, see (1). Damping can also be used for numerical stability by scaling down the larger spectral coefficients (m, n), see (6).

A forcing term can account for heating at the equator, see (1). Also, forcing can counteract damping by adding vorticity varying in space but constant in time in a way appropriate for the context, see (6).

3.2 Parallelization

The author proposes computing the model in parallel by assigning each layer to one computing device, and for each time-step, computing the layers in parallel, then coupling the layers. When this process is repeated, the system of equations can be computed for an arbitrary number of time steps.

The algorithm run-time is linear, $O(N)$, in the number of layers N, since each layers is coupled with only its two neighbors. So at first guess, one might expect that doubling the number of processors will halve the run-time. This is not the case because part of the algorithm is not computed in parallel. So the speedup is subordinate to Amdahl's law which says the computation time $T(p)$ using p computing devices is

$$T(p) = T(1)\left(\text{fraction of serial run-time} + \frac{1 - \text{fraction of serial run-time}}{p}\right).$$

To maximize speedup, $\frac{T(1)}{T(p)}$, we must minimize the quantity in the parenthesis. This is done by minimizing the serial run-time. A second look at the quasigeostrophic model shows that the serial vertical coupling part is also parallelizable. That is, the vertical coupling is performed for each spectral grid point (m, n), so the spectral grid points can be partitioned and distributed evenly to the computing devices. The modified Amdahl's law where the serial part is also done in parallel is

$$T(p) = T(1)\left(\frac{\text{fraction of serial run-time}}{p} + \frac{1 - \frac{\text{fraction of serial run-time}}{p}}{p}\right).$$

Some numerical experiments can now be performed.

3.3 Numerical Experiments

The author's advisor provided a serial implementation of (6). The code was parallelized by the author. Barriers were used before the per-layer part and before the vertical coupling part.

Eight height levels were used, since that is the maximum number of processors with shared memory available to the author. The serial run-time was benchmarked, and the fraction of serial run-time was about 0.1. So by the modified Amdahl's law, theoretical maximum speedup for eight processors is 7.36. The table below shows experimental and theoretical run-times.

TABLE I

RUN TIMES FOR 9600 TIME STEPS USING TRUNCATION $M = 21$ AND EIGHT HEIGHT LEVELS.

Number of processors	time (seconds)	Experimental speedup from 1 thread	Theoretically maximum speedup using Amdahl's law
1	10.60	-	-
2	5.36	1.98	1.90
4	2.83	3.75	3.72
8	1.52	6.97	7.36

The anomaly of two and four processors running faster than the theoretical max was perhaps due to the incorrect measurement of serial run-time. The relative slowdown for eight processors is due to the operating system scheduling interrupts and context switches, and the time when some processors are waiting at the barrier adds to the fraction of serial run-time. Context switches were minimized using “busy-wait” barriers. Interrupts were minimized by instructing the operating system to isolate those eight processors from scheduler contention. But the available schedulers were all known to require at least some interrupts and context switches, which caused the run-time to be less than the theoretical maximum.

3.4 Conclusion

The quasigeostrophic model, derived in chapter 2, was discretized, parallelized, and computed. Speedups were reasonable.

CITED LITERATURE

1. Pedlosky, J.: Geophysical Fluid Dynamics. Springer-Verlag, 1987.

2. Gill, A.: Atmosphere-ocean Dynamics. Academic Press, 1982.

3. Cushman-Roisin, B. and Beckers, J.: Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects. Academic Press, 2011.

4. Batchelor, G.: An Introduction to Fluid Dynamics. Cambridge University Press, 2000.

5. Kundu, P. and Cohen, I.: Fluid Mechanics. Elsevier Science, 2010.

6. Marshall, J. and Molteni, F.: Toward a dynamical understanding of planetary-scale flow regimes. Journal of the Atmospheric Sciences, 50(12):1792–1818, June 1993.

7. Haltiner, G. and Williams, R.: Numerical prediction and dynamic meteorology. Wiley, 1980.

8. Holton, J.: An Introduction to Dynamic Meteorology. Elsevier Academic Press, 2004.

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